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The MathMate

The Official Journal of the
South Carolina Council of Teachers of Mathematics

Mission Statement: The mission of THE MATHMATE is to feature articles about innovative mathematical classroom practices, important and timely educational issues, pedagogical methods, theoretical findings, significant mathematical ideas, and hands-on classroom activities and disseminate this information to students, educators and administrators.

THE MATHMATE, the official journal of the South Carolina Council of Teachers of Mathematics, is published online three times each year – January, May, and September.

Submission Requirements: Submissions for THE MATHMATE should be no more than 15 pages in length not counting cover page, abstract, references, tables, and figures. Submissions of more than 15 pages will be reviewed at the discretion of the editorial board. Submissions should conform to the style specified in the *Publications Manual of the American Psychological Association* (6th ed.). All submissions are to be emailed to scmathmate@gmail.com as attachments with a completed Submission Coversheet as page 1 and the article starting on page 2. Click here to download THE MATHMATE Submission Coversheet.

Submitted files must be saved as MSWord, RTF, or PDF files. Pictures and diagrams must be saved as separate files and appropriately labeled according to APA style. Copyright information will be sent once an article is reviewed but authors should not submit the same article to another publication while it is in review for THE MATHMATE.

Submission Deadlines: Submissions received by October 1 will be considered for the January issue, February 1 for the May issue, and June 1 for the September issue.

South Carolina Recertification Credit: According to the SC Department of Education Renewal Credit matrix, http://ed.sc.gov/agency/se/Educator-Services/Licensure/documents/renewalmatrix2013.pdf, the primary author of a refereed journal article can earn 60 renewal credits.

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The MathMate

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Message from the SCCTM President

Dear Members,

Welcome to the first issue of THE MATHMATE for 2014! We are so excited to be able to get THE MATHMATE to our members after a lengthy delay in publication. We were overwhelmed by the response that we received from our members and their willingness to contribute. Thankfully, we now have a dedicated member, Gina Dunn, who has willingly volunteered to be the editor for THE MATHMATE.

I hope that you appreciate the time and effort that the writers, reviewers, and editorial board have spent in publishing this journal. Please feel free to use the information you gain from the articles in your classroom. You may copy the activities from the journal and use them in your classrooms and share them with your colleagues. We are already working on the next issue and you play a vital role in the success of THE MATHMATE. Please take the time to submit an article showcasing either a classroom activity or research project that would be beneficial to other mathematics educators. Information about submission of articles can be found in THE MATHMATE and on our website (http://www.scctmprogram.org) or you can send it directly to our editor, Gina Dunn, at SCMathMate@gmail.com.

We have just held a successful conference at the TD Convention Center in Greenville, SC. We are already in the planning stage for our next conference which will be November 6-7, 2014 in Myrtle Beach, South Carolina. As with this year's conference, we hope to offer even more interesting and helpful sessions, workshops, and guest speaker keynotes. I hope that you begin now to make plans to attend the next conference in Myrtle Beach and get others to attend with you. Our goal is to promote mathematics in our state and positively impact student achievement in mathematics.

Please encourage your colleagues to join the SCCTM and take advantage of the members-only privileges.

Sincerely,
Jennifer E. Wilson
SCCTM President

Announcements

Upcoming Deadlines:

SCCTM Teacher Grants due April 15, 2014

Upcoming Conference Information and Deadlines:

SCCTM 2014 Annual Fall Conference, Myrtle Beach, South Carolina, November 6 – 7
Speaker Proposal Form due April 15, 2014

Membership News:

Renew your NCTM membership online and designate *South Carolina Council of Teachers of Mathematics* for the affiliate rebate.

If you would like your announcement to appear in the next issue of THE MATHMATE, please email all information to SCMathMate@gmail.com by May 1, 2014. Announcements will be published at the discretion of THE MATHMATE Editorial Board.

Guest Editorial

What does SCCTM mean to me?

Bill Stevens

What does SCCTM mean to me? That's a difficult question for me to answer. I've been involved with the South Carolina Council of Teachers of Mathematics since its early days. I should first point out that I did not intend to be a teacher. I expected to be a chemist at the Charleston Naval Shipyard, where my father worked for many years. Hearing of a teaching position at Goose Creek High School in September of 1967, I applied because I needed to find work after finishing my math and chemistry degree at the College of Charleston. Little did I know, that second semester would define my employment life for the next 39 and a half years. That was the semester that I was given an Algebra I class when overcrowding in current classes necessitated creating a new one. Having spent a frustrating first semester teaching "lower" eighth grade math to four classes of disinterested students, the new algebra class was a ray of sunshine and allowed me to see the joy in teaching, something I recognized in the inspirational teachers I had in high school. I realized the joy that my teachers at the High School of Charleston must have felt. I had a class of eager learners, anxious to actually do their best and ready themselves for the next year. GCHS stopped at the tenth grade my first year, and added an additional grade each of the next two years. I got to move up with the students and began teaching chemistry, geometry, and finally Senior Math during the 1969-1970 school year, my third year there. I later learned that Senior Math would now be called precalculus, but that was before there was calculus in high school. I spent seven years at Goose Creek High and made many friendships with faculty and students that extend to this day.

After Goose Creek, I was employed by Charleston Count School District in the Science Office and then Mathematics Office of the Division of Instruction. Starting in the Math Office in 1977, near the founding of SCCTM, I learned about professional organizations and what they had to offer their members. Joining SCCTM in its early years allowed me to watch it grow. I remember when our membership first hit 1,000 and even topped 2,000 several years later. As MathMate editor from 1982 to 1985, I began keeping the mailing addresses of our members for printing mailing labels and sending out the issues. I continued maintaining the names and addresses as I became President-Elect, President, and Past President from 1985 to 1988. I was then given the Database Manager position that was created in 1988, a position I still hold about 25 years later. After twelve years with CCSD, nine years of which were in the CCSD Math Office, I became a teacher again at Summerville High School for fifteen years, at West Ashley High School for two years, and finally at Summerville High again for my final four years teaching. I'm in my sixth year of retirement, but not from SCCTM.

Being a member of the South Carolina Council of Teachers of Mathematics is, to me, being professional. *There's the short answer to what I was asked to write about.* I've learned a great deal from my association with SCCTM; new teaching methods, new professional friends to share experiences with and gain ideas from, new knowledge from workshops and sessions at the Fall Conference, to name just a few. I was always troubled when other teachers would tell me that they weren't renewing their membership because they weren't going to the conference. Attending the Fall Conference is but one part of being a member of SCCTM. Membership is much more than a line on a résumé or vita. It's contributing to the goals and mission of the council, thereby helping students learn and appreciate the beauty and intricacies we see in mathematics. I remember realizing on my own that the multiples of nine added up to nine when I first studied multiplication. Little did I know that I was gaining an appreciation for number theory at that early age. I'm always saddened when people tell me, sometimes proudly, that they were never good in math. If the opportunity arises, I point out that very few would admit the same thing about reading. I wonder why that is!

Membership in SCCTM and what you can gain from it far outweighs the small cost of a yearly membership. I know that active members in our organization will be the ones to read this in the revived MathMate, but maybe they can help SCCTM regain the members it's lost in recent years. As keeper of the database, I've seen our membership shrink as attendance in the Fall Conference became smaller and smaller. I'm happy to report that we're still over 1,000 active members, but we just got there the week before I completed this. SCCTM could provide so much more to its members and their students with a membership total back over 2,000. I'm sure there are still that many eligible math teachers in the South Carolina. How many of your fellow math teachers are also members?

The Pen Pal Journal Project

Jennifer Crooks Monastra
University of South Carolina

Abstract

As a part of mathematics content courses, elementary education majors at the University of South Carolina (USC) get a chance to practice teaching mathematics through a math pen pal program. Pre-service elementary teachers have a variety of feelings about teaching mathematics. Some declare it to be their favorite subject, but have never experienced the opportunity to put their excitement into practice. Others admit they never liked mathematics, and some university students express uncertainty about their ability to teach it. The authentic experience of the pen pal program gives pre-service teachers an opportunity to investigate their feelings about teaching mathematics, as they experience some joys and challenges of teaching.

Math pen pal journals are a written exchange between university students and students from local elementary schools. The university students are either elementary or early childhood education majors and are enrolled in a mathematics content course at USC. Elementary students from second grade through fifth grade have participated in the pen pal project. University students are paired with randomly selected elementary students.

Each set of pen pals designs a folder, with one side decorated by the USC student and the other by the elementary student. Many college students include a picture of themselves and stickers throughout the entries, while the elementary students often include their own drawings. The groups typically exchange about five rounds of letters. The pen pals begin building a relationship by introducing and sharing about themselves, often discussing pets or favorite school subjects. They ask and answer questions in each exchange.

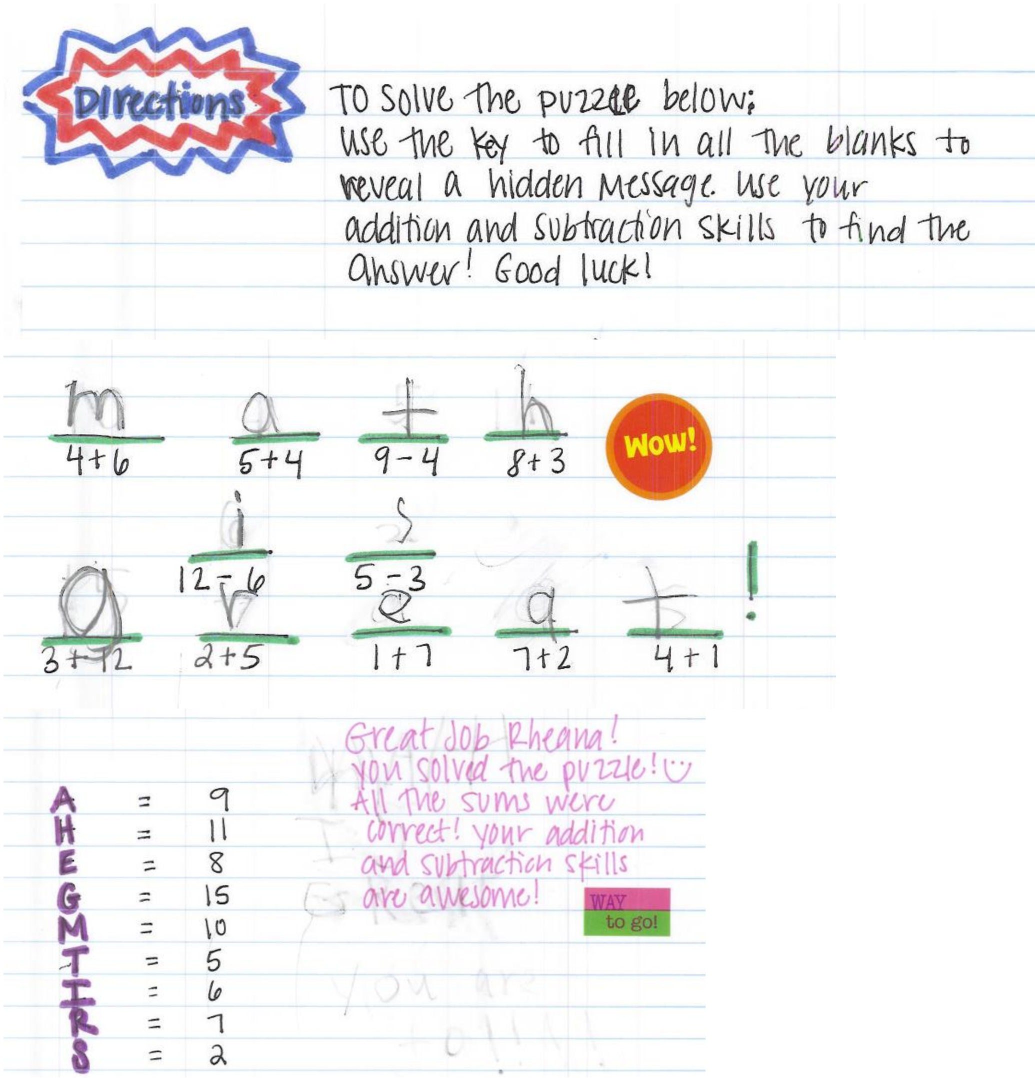
Dear Rheana,

Hey! My name is Macie and we'll be math pen-pals for the next few months! Math is my favorite subject. What is your favorite subject? I am 19 years old and am a student at the University of South Carolina. I want to be an Elementary School teacher when I get out of college.

I wrote a math problem on the next page. You can try answering it and write me a math problem back. You can also decorate the front of this notebook like I did on the back! I look forward to hearing back from you soon!

Your friend,

Macie!

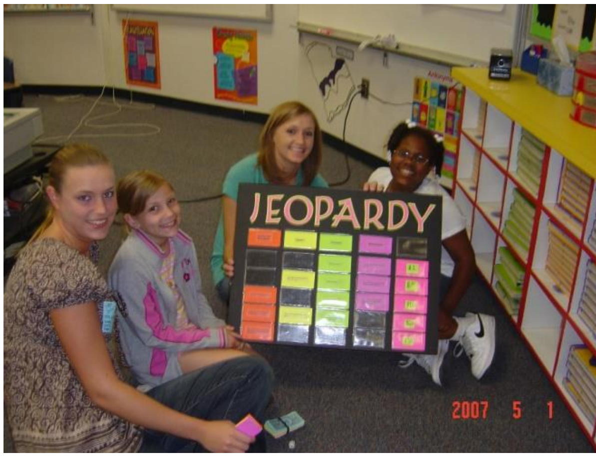


In addition to letter writing, the pen pals send math problems to each other. The problems begin with one or two examples with explanations. This is followed by one or two problems for their pen pal to work out and explain. The USC pen pals do not send purely computational problems, but focus more on applying concepts, using real world situations when possible.

The mathematical content for the journal entries is based on the Common Core Curriculum Standards (CCCS) and can be chosen in different ways. One option is for pre-service teachers to choose any grade-level appropriate mathematics standard they want to teach. Another option is to have the problems in the journals relate to the content being taught by the elementary school classroom teacher. The journals could be used by the elementary school teachers to strengthen and deepen their students' learning. This requires good communication between classroom teacher, university instructor and pre-service teachers, to keep journal entries current with classroom teaching.

Sometimes the journals stay on the same standard for several weeks. This can happen when the initial level of difficulty for the problem does not match well with the child's ability level. An elementary student may struggle with the topic but slowly make progress. Sometimes, the elementary students ask for harder problems. The pre-service teachers learn to adapt their questions to their pen pal's abilities, and to incorporate content that is interesting to both themselves and their pen pal. In this way, each child gets a customized journal.

The project concludes with a pen pal party, where the pre-service teachers get to meet the elementary students. The USC pen pals prepare a math game to teach and play with the elementary students. Sometimes this is a familiar game, such as bingo or jeopardy, with math facts for the questions. Other university students use ideas from websites or invent their own games. Many design their own game boards. The games are left with the elementary school students, to be taken home and played with their families.



Benefits for elementary students

The math pen pal journals provide several academic benefits for students. The journals are an example of implementing the valuable practice of writing across the curriculum (James R. Squire Office of Policy Research, 2011). The pre-service teachers encourage their pen pals to explain their mathematical thinking. Many times students can give the numeric answer to a problem, but asking for an explanation takes their thinking to a deeper level. It also helps transform mathematics in the eyes of students, from a list of random rules to be memorized and executed to a meaningful discipline. In addition, the writing itself provides students with an opportunity to practice their grammar, handwriting, and communication skills.

Another benefit of the journals is connecting elementary students with higher education. I am surprised at the respect and admiration elementary students have for university students. Some are amazed a “real college student” is writing to them. They ask their USC pen pals about their classes and teachers. Many of the elementary children look up to their pen pals, and are thrilled to receive the individual attention of a college age adult. Other elementary students are excited their pen pals want to be teachers, and know they are helping future teachers practice for their profession. The journals allow the younger students to dream and think about their future, and potentially build their interest in higher education.

One third grade teacher described the project as “every teacher’s dream – to have students excited about writing and math,” and the participants described the project as fun. The elementary teachers reported a new level of excitement in their classroom every time the journals arrived. I have also done this project with my daughter’s elementary school class. A year later, she tells me that some of the students from her class still talk about the project, and ask if they can do it again

Benefits for pre-service teachers

For many university students, this is their first opportunity to try and formally teach any educational content to an elementary student. They are enthusiastic about getting to be the teacher. This authentic practice brings a host of emotions including surprise, frustration, and excitement. It is challenging for them to put their own mathematical understanding into writing. Writing causes them to think more critically about the vocabulary they use, and gives them concrete practice at creating their own problems.

Pen pal journals provide the pre-service teachers practice giving written feedback to students which is an important pedagogical skill (Stronge, 2007) and requires practice to learn. Several university students were unsure how to respond when their pen pal’s work was incorrect. They ask questions like, “My pen pal got this problem wrong. What do I do?” Being the one to provide a response forced them to think like a teacher, respond to student errors, and develop their own mathematical communication.

The pen pal responses also provide insight into how children think. Many elementary students send problems for their pen pals to work. Sometimes they use large numbers with lots of zeros at the end. Younger children seem to believe that more digits make problems more difficult. Sometimes it is difficult to understand what the elementary student is trying to ask. This leads to interesting discussions about how students use their mathematical knowledge.

In addition, pre-service teachers learn about the CCCS and can see connections to their college coursework. Having pre-service teachers engage in researching the standards, increases their awareness of the content in elementary school mathematics.

What the pre-service teachers learned

1. It can be difficult to explain a concept that you understand. Many students expressed surprise at how challenging it was for them to explain why, and how, they solve a problem. They can solve the problem, but explaining it to others is a different skill.
2. Students may not respond in the manner you expect. Sometimes the university students were surprised by the responses the elementary students sent back. The elementary students approach was different from their expectations. This made the pre-service teachers consider the validity of other approaches, which is a valuable pedagogical skill.
3. Elementary school age children's abilities may not match expectations. Some students were amazed at how "smart" their pen pals were. They would comment on having to raise the level of difficulty in the problems they were sending since their pen pals got it all right, or sometimes said "this is too easy". Other times they were surprised that their pen pals could not solve their problems. When students had the opportunity to view others' journals, they noticed the range in their pen pals' abilities. Some elementary students could send a page long letter, solve the math problems, explain their thinking and send problems for their pen pal to solve. Others seemed to struggle with handwriting, would only write a few sentences or not attempt their math problems. This gave the pre-service teachers exposure to the range of abilities students may have in their future classrooms.
4. Teaching is fun! Getting to see someone grasp a new concept is a great benefit of teaching. After a pen pal party, one USC student was beaming because she had experienced teaching her pen pal a concept from multiplication. The opportunity to build a relationship with a child was also valuable. After meeting their pen pals, several university students commented that their experience helped to confirm that they had chosen the right major.

Recommendations for implementation

Logistical challenges may surface when implementing this project. Arrangements must be made for transporting journals back and forth in a timeframe conducive to both elementary and university scheduling. This requires good communication and planning, as well as flexibility throughout the process. Student absences provide another challenge. Preservice teachers can be very disappointed if they do not receive a response from their pen pal. Occasionally pen pals need to be reassigned, for instance when a student moves to another school. The benefits to the participants seem to outweigh these challenges.

It will not always be possible for elementary schools and universities to work together. However, this project could be implemented in other ways. For instance, pen pals could be formed between a middle school and an elementary school, or between parents and children. One school I worked with was so disappointed when the project ended, that they chose to continue journal correspondence on their own. They had two classes at the same grade level begin writing letters to each other. As Bochese (1997) describes, it can also be used to connect diverse populations by writing between schools whose students differ in ethnicity or economic status.

References

Bochese, J. (1997). Math Pals Write to Solve. *Teaching Children Mathematics*, 3, 488-489.

James R. Squire Office of Policy Research. (2011, March). Reading and writing across the curriculum. Council Chronicle, 20(3), 15–18.

Stronge, J. H. (2007). *Qualities of effective teachers*. Alexandria, VA: Association for Supervision and Curriculum Development, 85-98. (Chapter 4: *Monitoring Student Progress and Potential*)

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What Our Students Have to Say… *Students'* Favorite Lessons

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Indian Land High School

Abstract

This article leans on the expertise of high school students. They share what their favorite lessons were in algebra, geometry and calculus.

Three high school students were invited into a preservice mathematics teacher methods course to offer their views of good mathematics teaching to soon-to-be mathematics teachers and their instructors. We were all ears. Few other lessons in the course had commanded such attention. What did current high school mathematics *students* think was good mathematics teaching? What were *their* favorite lessons?

Algebra: Fly a Kite

One student remembered constructing a kite in class and then taking it outside to put his construction to the test in a flight. Among other concepts, he used ideas of mathematical modeling and function to represent relationships between area and perimeter (NCTM, 2000). Here is one sample lesson: http://www.nsa.gov/academia/_files/collected_learning/middle_school/geometry/go_fly_kite.pdf. NASA also has a simulator that teachers can use in their classroom: http://www.grc.nasa.gov/WWW/K-12/airplane/kiteprog.html.

Algebra: Rockets as Parabolas

Rockets provide models in multiple courses. These students remembered launching a rocket on the football field to model parabolas in algebra. Weiss et al. (2002) used rockets to address AP Calculus objectives.

Geometry: Build A Town

New Common Core State Geometry Standards (CCSSI, 2010) asks students to use geometric methods to solve design problems (e.g. Kasprzak, 2002). One student fondly remembered the lesson assignment to build a town in geometry. She remembered she was asked to use a certain number of geometric elements such as angles, arcs, polygons, and polyhedra.

Geometry & Physics: Construct a Bridge

"We built these bridges and then went to the weight room to see how much weight they could hold. I remember one bridge held one of the smaller kids!" A similar lesson can be found here: http://www.engineeringplanet.rutgers.edu/pdf/lessons/engineering/civil_enviromental/2004/lesson10.pdf. A similar lesson involving paper and pennies can be found here: http://www.gk12.msstate.edu/lessonplans/039_INSPIRE_LP_4_Nash_09_01_10.pdf.

Calculus: Design a Roller Coaster

"I remember the roller coaster we built in calculus." This project can be done solely from a theoretical perspective such as here: http://wp-blogs.moundsparkacademy.org/apcalc/files/2011/08/CoasterProject20111.pdf or teachers can take it a step further, which is what this student remembered. One other resourceful website for a roller coaster project is http://www.mastermathmentor.com/calc/calc-projects.ashx. It requires a login but is free to use.

Concluding Remarks

Hearing these students relive their most exciting lessons reinforced for us the importance of connecting mathematics to their lived experiences. We were also reminded of how mathematical lessons such as these incorporate multiple content areas and reinforce to students that mathematics, although often organized by domains such as algebra, geometry and calculus, is a connected language. This exercise of asking students what lessons they enjoyed most reminded us that the most important factor in mathematics classrooms are the students…and we should ask them what they think more often than we do.

References

Common Core State Standards Initiative (2010). *Common Core State Standards for Mathematics.* Washington, DC: National Governors Association Center for Best Practices and the Council of Chief State School Officers. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf

Kasprzak, E. M. (2002). Design a window. *Mathematics Teacher, 95*(5), 346-353.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and standards for school mathematics.* Reston, VA: Author.

Weiss, M. (2002). Using a model rocket-engine test stand in a calculus course. *Mathematics Teacher, 95*(7), 516-519.

Authors

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Dr. Patenaude is a National Board Certified teacher at South Pointe High School in Rock Hill, SC. He recently earned his PhD from the University of South Carolina completing his dissertation on using applets for developing understanding in calculus.

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Bracelet Links: Division by a Fraction

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Brooke Lance
Lakeside Middle School

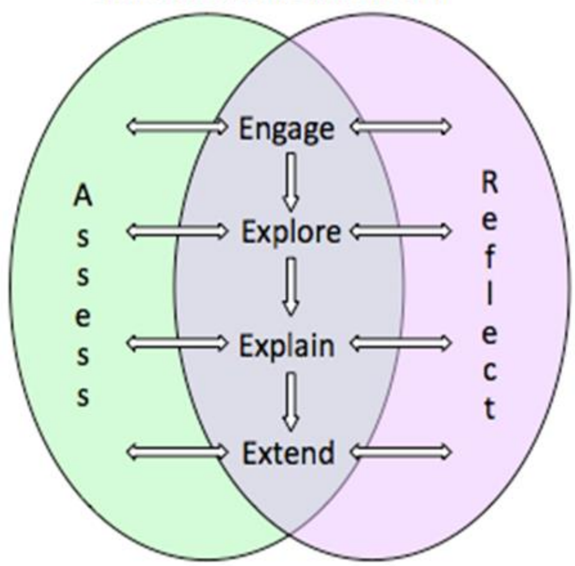
Abstract

Division by a fraction is one of the most poorly understood algorithms. This article explores students' conceptual understanding of division by a fraction when an inquiry-based approach is followed and when a color-screen graphing calculator is used as an instructional tool.

As math teachers, we know that problem-based approaches in mathematics, coupled with the appropriate use of technology, can deepen a student's understanding. When graphing calculators are used in problem-based situations, students can create, explore, and analyze models to help them make important mathematical connections. Graphing calculators can present students with accurate and multiple representations, thus creating more time for reflection and discourse. Technology can also reduce students' cognitive load so they can focus on underlying concepts, not on carrying out poorly understood algorithms (Clarke, Ayres, & Sweller, 2005). With the aid of graphing calculators, students can actively and constructively develop new knowledge and understanding.

The Common Core State Standards for Mathematics (2012) identifies eight Standards for Mathematical Practice that every mathematics teacher should endeavor to develop in their students. When the Standards for Mathematical Practice are joined with the content, inquiry-based instruction is possible (Marshall, Horton, & Smart, 2009). Marshall, et. al. (2009) have developed a model that focuses on this inquiry-based approach. Their 4E x 2 (read "4 E by 2") construct brings together inquiry, assessment, and reflection while providing a structure for mathematics inquiry-based instruction. When this model is married with the idea of using technology as a learning tool, students can begin to develop rich, conceptual understandings (see **Figure 1**).

Figure 1: Framework for the 4E x 2 Instructional Model



Specifically, the 4E x 2 model helps teachers' guide students through a particular learning cycle, continuously assessing student learning and reflecting on progress. This type of structure helps ensure that students have time to participate actively in developing conceptual understandings of topics by allowing them to explore ideas *before* explanations are provided. The investigation presented in this article follows the stages of the 4E x 2 model: Engage, Explore, Explain, and Extend. While it may appear that this learning structure is linear in nature, it is intended to be dynamic, allowing students to develop an understanding of topics by revisiting particular stages when needed (Marshall, Horton, & Smart, 2009). The "by 2" part of the model refers to the formative assessment and teacher reflection that should accompany each stage of the inquiry. By gathering information through assessments, teachers can make intentional decisions as to whether students are ready to proceed, need remediation, or perhaps can be accelerated.

In the "Engage" stage, teachers should not only provide a motivating question to garner students' interest, but they should also check for prior knowledge and probe for misconceptions. In the "Explore" stage, students should be given the opportunity to investigate the ideas on their own or in small groups prior to an explanation of the content. The "Explain" stage should follow the "Explore" stage, with both teachers and students contributing. Teachers should keep in mind that the answer to the question at hand is not the important part of the lesson; instead, the teacher should develop and emphasize the underlying ideas. Lessons will often have multiple "Explore-Explain" cycles, sometimes lasting only a couple of minutes and at other times carrying over into several class periods. The "Extend" stage deepens the learning of the underlying ideas, and often takes the form of additional Explore-Explain stages. For detailed information about the 4E x 2, visit www.clemson.edu/iim. A webinar explaining the model in detail, a lesson planning repository, and a lesson planning tool are all available.

A Partnership

In 2010-2011, Lakeside Middle School in Anderson School District 5 partnered with Clemson University in an effort to promote inquiry-based instruction. Four of the six members of the middle school mathematics department (along with the entire science department) participated in an intense, two-week summer institute designed to allow teachers to experience inquiry-based instruction, to perturb them about their instructional practices, and to develop inquiry-based lessons for their classrooms. The teachers received individual support throughout the school year and participated in several follow-up meetings. These efforts were supported through grants from the State Commission on Higher Education and the National Science Foundation.

This article details an investigation, Bracelet Links, the math teachers used to promote students' understanding of division by a fraction. The investigation required that students rely on modeling with the aid of a color-screen graphing calculator, in this case the Casio PRIZM. Specifically, students were engaged in creating diagrams to represent number sentences; students were involved in understanding the relationship between a diagram of a bracelet, a number sentence and, more specifically, the relationship between the divisor, dividend, and quotient. Furthermore, they analyzed "those relationships mathematically to draw conclusions. They interpret[ed] their mathematical results in the context of the situation and reflect[ed] on whether the results [made] sense" (CCSSM, http://www.corestandards.org/Math/Practice/MP4).

Bracelet Links: The Problem

Operations with fractions, especially division by a fraction, are often understood procedurally but not conceptually. In fact, inverting and multiplying is one of the most poorly understood algorithms (Van de Wall, 2007). The rules and procedures for fraction operations do not help the student think about the meaning of the operations. If we want students to understand fractions and fraction operations conceptually, we must be intentional in how we frame and structure questions and learning experiences.

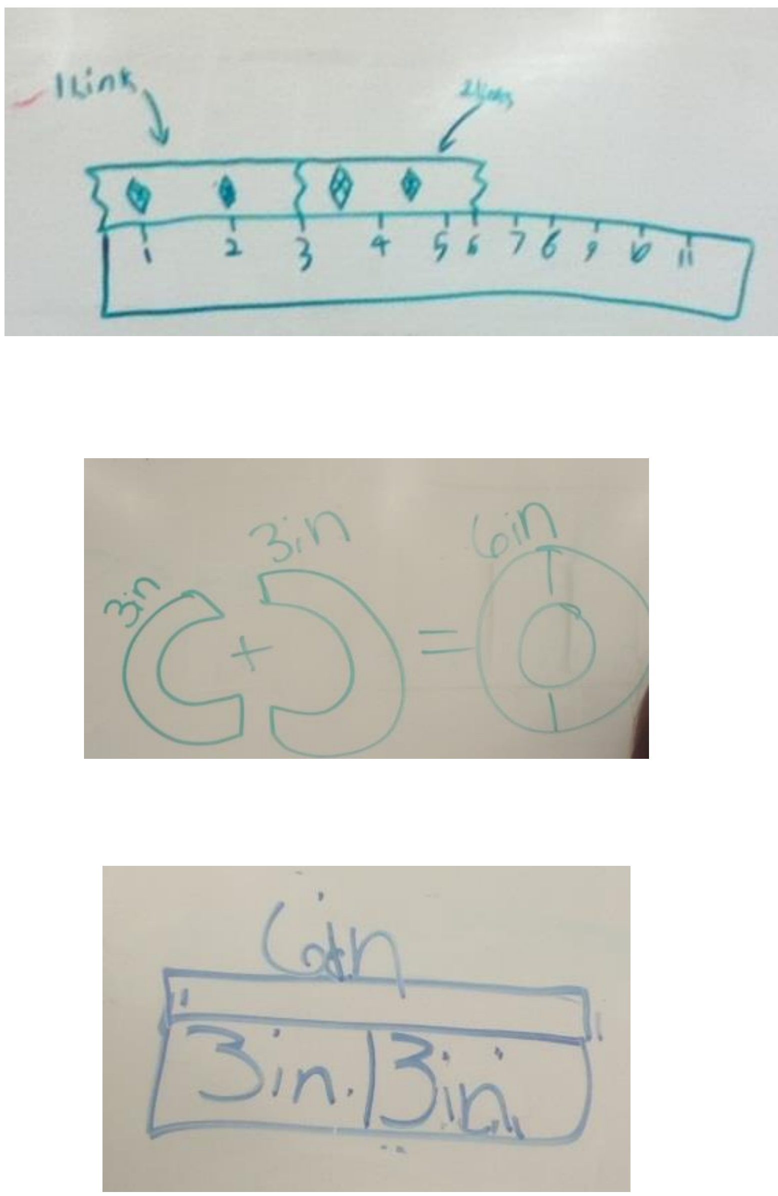
To help students engage in a meaningful mathematical task associated with dividing by a fraction, the following problem was presented to 7^{th} grade students, an extension of a focus on developing fraction ideas in 6^{th} grade:

> For an art project, your class will make bracelets by linking different pieces together. Each bracelet must be 6 inches long. For this investigation, you are to explore the relationship between the length of each link and the number of links required to make the bracelet.
>
> A. If each link is three inches long, how many links will be required to make the bracelet? Draw a model to represent your thinking and represent the ideas in symbols.

Engage

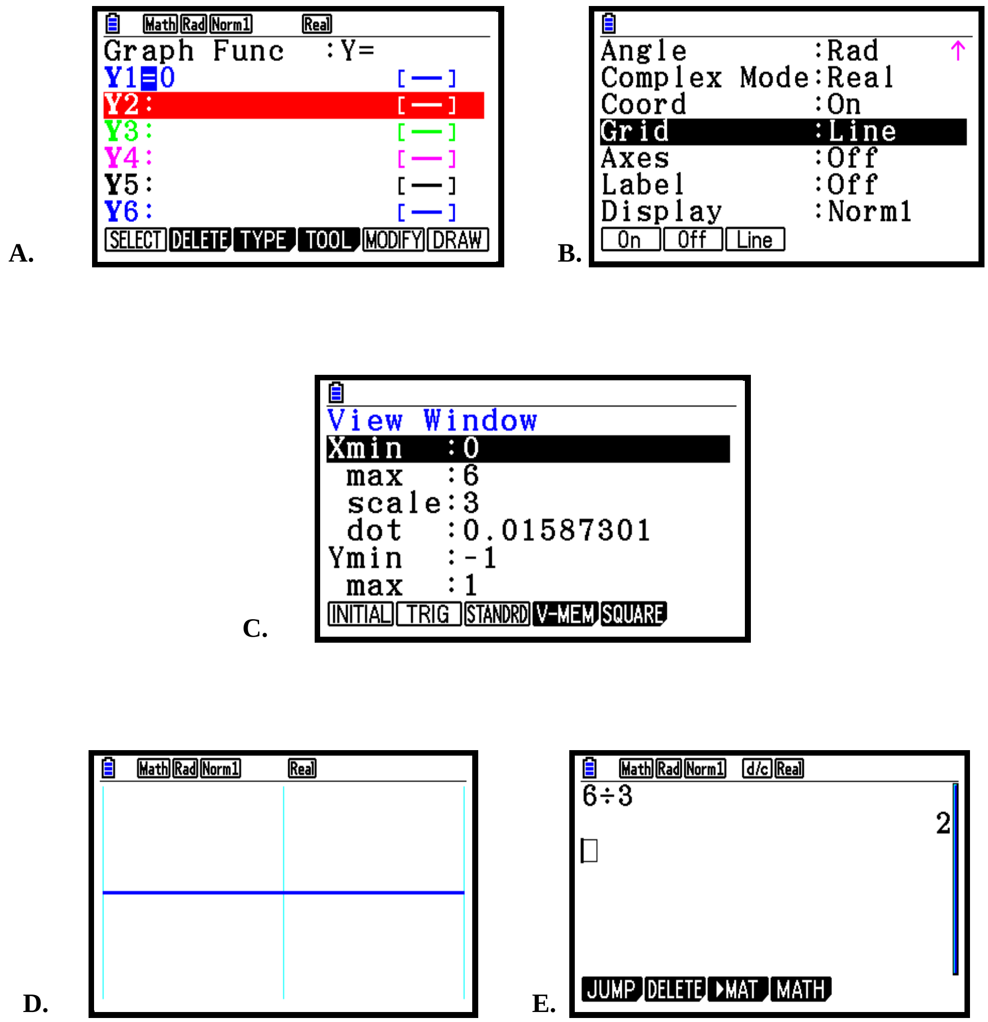
The meaning of division of fractions is the same as the meaning of division by whole numbers (Van de Walle, 2007). Thus, to begin, the teacher engaged students in the investigation by presenting the problem, first focusing attention on the meaning of whole number division, building on the students' prior knowledge and current conceptual understanding of this operation. In this particular context, division represents repeated subtraction. After presenting the problem, the teacher encouraged students to consider multiple drawings to represent the division problem and asked students to explain their thinking. Students were also expected to think about the operation that represented the problem. After individual work time was provided, several students were asked to draw their model, share it with the class, and explain how their thinking related to their model (see **Figure 2**). Data indicated that students understood the context and could represent the problem pictorially. Discussion also indicated that students knew how to use division to represent the problem and its solution. However, during the discussion, when we asked students if the quotient is always smaller than the dividend (in this first problem, the quotient 2 is smaller than the dividend 6) we found that most students believed it is, that division "makes smaller." We then knew we would have to confront this misconception directly as we progressed through the problem.

Figure 2



After spending time exploring various pictorial models, the teacher introduced the graphing calculator as a tool to aid in developing a concrete understanding of dividing by a fraction. The teacher guided students in representing the bracelet by constructing a horizontal line, specifically with the line y=0. With teacher guidance, students were instructed in setting the domain from 0 to 6 to represent the six-inch bracelet and the scale at 3 to represent the length of each link. In this initial problem, students considered how many links would be required to make a bracelet of six inches if each link is three inches (see **Figure 3**).

Figure 3

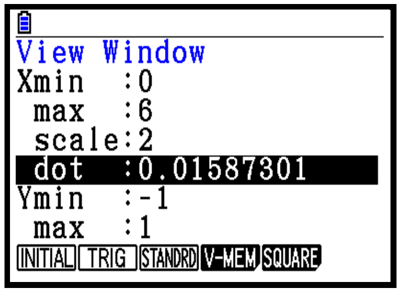


- The bracelet is displayed as the equation y = 0. **(A)**
- The grid was turned on, dividing the bracelet into varying lengths. **(B)**
- The scale was adjusted to three, representing the length of each link. **(C)**
- The graph represents two links that are put together to make a bracelet of 6 inches. **(D)**
- The graphical representation is connected with the expression 6÷3. **(E)**

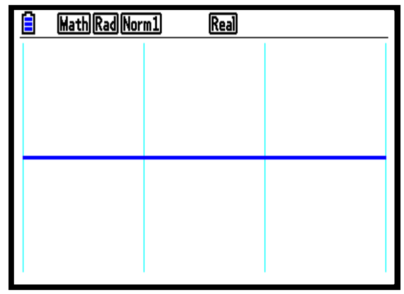
Students continued to use their graphing calculators to engage in the meaningful task of understanding division. They next considered:

B. If each link is two inches long, how many links will be required to make the bracelet? (see **Figure 4**)

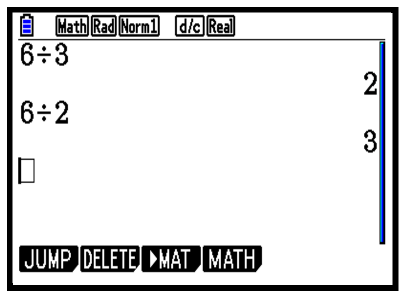
Figure 4



A.



B.



C.

- The bracelet is again displayed as the equation y = 0.
- The viewing window had to be adjusted to display links that are two inches; thus the scale was adjusted to 2. **(A)**
- The graph represents three links that are put together to make the bracelet. **(B)**
- The graphical representation is connected with the expression 6÷2. **(C)**

Additionally, the teacher's questions helped foster an atmosphere of exploration and inquiry.

- How does this graphical model represent our bracelet?
- From this model, how can we tell how long the bracelet is?
- From this drawing, how can we tell how long the links are?
- How would we write a number sentence to represent this problem?
- In what ways does this number sentence relate to our drawings?

Explore

During the Explore phase of the learning cycle, students were problem-solving, representing their thinking through multiple-representations, connecting their representations to the problem and to the number sentence, and communicating their ideas. These processes helped students construct meaning. The graphing calculator aided students as they explored the following problems:

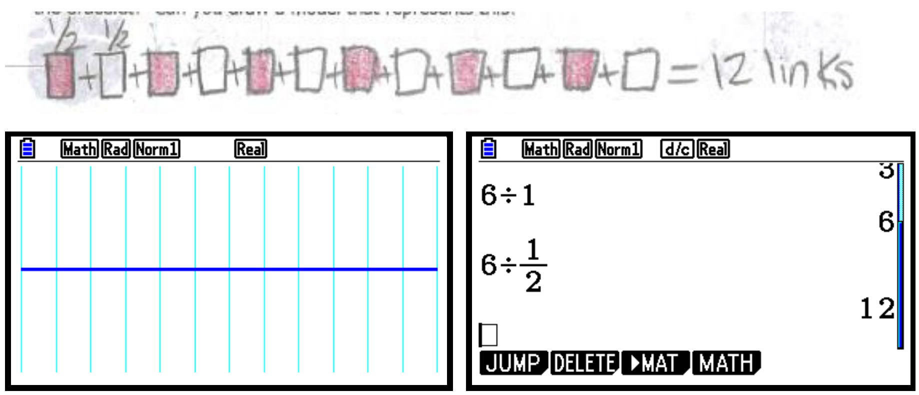
C. If each link is one-inch long, how many links will be required to make the bracelet?
D. If each link is one-half inch long, how many links will be required to make the bracelet?
E. If each link is one-third inch long, how many links will be required to make the bracelet?
F. If each link is one-fourth inch long, how many links will be required to make the bracelet?

Students again used features of their calculator to represent their thinking in each scenario. Connections were made between the problem, their pictorial model, and their number sentence; more importantly the calculator assisted students in making the connection between division by a whole number and division by a unit fraction. With each new prompt, students were first asked to predict how many links would be needed to make the bracelet, explore the situation graphically, and then solve the problem and justify their solution. Students voiced their thinking in small groups and with the whole class by sharing their calculator screens for each problem and by explaining their thinking. The teacher's questions continued to guide and support student thinking as they made connections between the different representations of the problem.

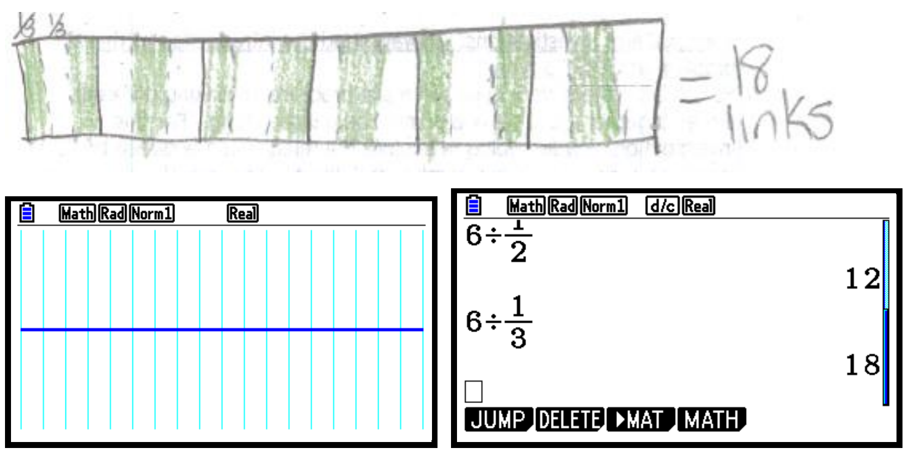
Students were able to make the leap to division by a unit fraction in Part D easily by building from Part C. Once they understood why six links, each measuring one inch, were required to make a bracelet totaling six inches, they could break each one-inch link into two pieces to understand why it would take twice as many links of one-half inch to make a bracelet of six inches (see **Figures 5**).

Figure 5

A. If each link is one-half inch long, how many links will be required to make the bracelet?



B. If each link is one-third inch long, how many links will be required to make the bracelet?



Explain

During the Explain stage of the 4E x 2 learning cycle, students' attention was focused on the content of the problem, division by a fraction, though to this point they had focused only on dividing by unit fractions. It is during this phase that the processes used while exploring ideas were formally connected to an algorithm. Students analyzed and interpreted their models and developed an algorithm to represent those models, cementing their understanding.

In order for students to develop an algorithm for dividing a whole number by a unit fraction, it was critical that they were allowed to explore the problem prior to the explanation (this is the heart of inquiry), with the teacher guiding and the students contributing significantly to the explanation of the algorithm they were developing. Doing this minimizes teacher-centered learning, which is often superficial, and inspires student-centered learning (Marshall, Horton, & Smart, 2009). Again, effective questioning guided students through this process and student responses to questions helped the teacher realize that students had internalized a solid understanding of the operation, answering questions such as “What is 8 divided by 1/6? Why?”

At this point, we also confronted the misconception about division we had unearthed during the Engage phase. We asked students why the result (the quotient) could be larger than the original number (the dividend); though we reinforced this later on, they responded that when the length of the links was less than 1 inch, it would take more than 6 of them to create the bracelet.

Extend

Students were allowed to extend their understanding of division by a fraction by considering how many bracelet links would be required if each bracelet link was not a unit fraction. The problem was also adapted to help students consider how many links would be needed if the total length of the bracelet changed. The next two prompts addressed these ideas:

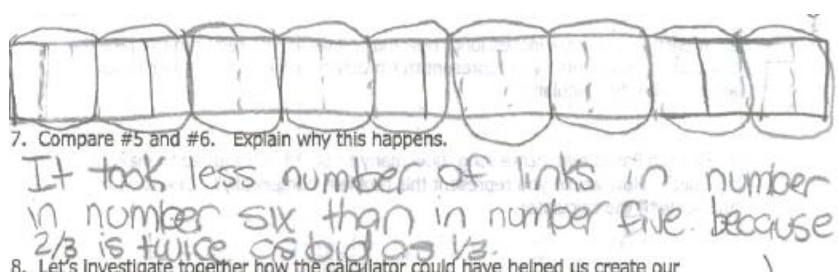
C. If each link is two-thirds of an inch in length, how many links will be required to make a bracelet of six inches? Represent this problem with a drawing and using mathematical symbols.

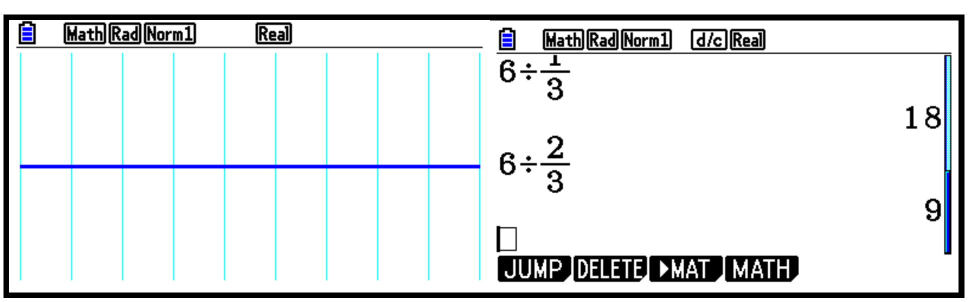
D. If the bracelets are to be 16 inches long, how many links would be needed if each link is to be two-thirds of an inch in length? Represent this problem with a drawing and with an equation. Explain why your result makes sense.

Because students had multiple opportunities to relate their knowledge in meaningful and authentic contexts, they were able to solidify their conceptual understanding, allowing them to develop a more permanent mental representation (Marshall, et. al., 2009). Students summarized their ideas about division of a whole number by a fraction and began to understand the algorithm intuitively (see **Figure 6**).

Figure 6

If each link is two-thirds of an inch in length, how many links will be required to make a bracelet of six inches?





Conclusion

Uniting reflection and formative assessment with inquiry provides teachers with a model to focus instructional practices on issues that improve teaching and learning (Marshall, Horton, & Smart, 2009). By using a problem-solving setting and using the graphing calculator as a tool, students developed for themselves the underlying reason for the "invert and multiply" algorithm, though to this point their experiences were limited to whole number dividends. Through modeling and discussion, they also found for themselves that the quotient, in our context the number of needed links, is larger than the dividend when the divisor is less than 1. Not only are these critical mathematical ideas, but students also began to realize that they can think and explore for themselves, that they indeed have significant mathematical power.

Students of course will need reinforcement of what they've learned, but they now have a context and mental image that helps them understand division by a fraction. Did the calculator help them understand? When we asked them, the overwhelming majority agreed that it did.

To conclude the lesson, we left the students with a challenge: How many links would it take if the bracelet was to be 15 inches long and each link is 2/3 inches?

References

Clarke, T., Ayres, P., & Sweller, J. (2005). The impact of sequencing and prior knowledge on learning mathematics through spreadsheet applications. *Educational Technology Research & Development, 53*(3), 15-24.

Garofalo, J., Drier, H., Harper, S., Timmerman, M. A., & Shockey, T. (2000). Promoting appropriate uses of technology in mathematics teacher preparation. *Contemporary Issues in Technology and Teacher Education, 1*(1), 66-88. Retrieved from http://www.citejournal.org/vol1/iss1/currentissues/mathematics/article1.htm

Marshall, J., Horton, B., and Smart, J. (2009). 4E x 2 instructional model: Uniting three learning constructs to improve praxis in science and mathematics classrooms. *Journal of Science Teacher Education, 20*(6), 501-516.

National Council of Teachers of Mathematics (NCTM). (2000). *Principles and Standards for School Mathematics.* Reston, VA: NCTM.

Van de Walle, J. (2007). *Elementary and Middle School Mathematics: Teaching Developmentally.* 6th ed. Boston, MA: Pearson Education.

Marshall, J. C., and Horton, R. M. (2009). *Developing, Assessing, and Sustaining Inquiry-Based Instruction: A Guide for Math and Science Teachers and Leaders.* Saarbrucken, Germany: VDM Verlag Dr. Muller.

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Alternative Ways to Find a Tangent Equation to a Curve

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Abstract

This article presents a very fast and effective analytical way to find the equation of a tangent line to a curve. The first method shown can be applied in finding a tangent line to a given conic, while the second one can be used in a scenario involving a higher-degree curve. This technique does not require calculus skills; therefore it can be used at virtually all high school and college math levels.

Normally, the evaluation of tangent equations at a curve is only presented at a higher level of math in the high school or college curriculum, such as the Calculus AB or Calculus BC courses and involves the knowledge of derivatives. Due to its simplicity and versatility, this technique can be presented to students in Geometry, Algebra 2 and Pre-Calculus related courses and it can be applied virtually anywhere a tangent line equation is needed.

In simple words, a tangent line to a curve at a certain point of the curve is a straight line that only touches the curve at the given point, without cutting it. Consequently, the tangent line would actually be the instantaneous linear behavior of the curve at a given point. For example, if you were to drive a car on a road that did not have a linear shape and suddenly you were to hit a very slippery spot on the road (water, oil, etc.), your car would follow the direction of the tangent line at the road curve at that slippery point. Also, if you were to swing an object in a circular motion then release it, the object would follow the trajectory of the tangent line to the circular motion path at the point of launch. Other practical applications of a tangent to a curve can be found anywhere an instantaneous rate of change is needed. For example, in order to find the instantaneous velocity of an object thrown upward at a certain time, you would have to calculate the rate of change of its distance vs. the time elapsed from its launch. This is the place where the tangent equation becomes useful, since the tangent equation has the same slope as the quadratic equation of its position vs. time at any given point.

Method I: Finding the equation of a tangent to a conic through one of its points:

According to a theorem found in *The Elements of Analytic Geometry* (Franklin & Sullivan, 1904) on page 212, the equation of a line tangent (t) to the locus of a curve (c): $Ax^2+By^2+Cxy+Dx+Ey+F=0$ at a point (x_o,y_o) on the locus is

(t) $A\,x\cdot x_o + B\,y\cdot y_o + C\,\frac{x\cdot y_0+y\cdot x_0}{2} + D\,\frac{x+x_0}{2} + E\,\frac{y+y_0}{2} + F = 0$

The full proof of this theorem can be found on page 213 of the same book and it starts by finding the equation of a line going through two points $A(x_o,y_o)$ and $B(x_o+h,y_o+k)$ on the curve (c), where k/h will serve as the slope of the secant AB. The equation of the tangent line is determined by the equation of the line AB when h and k approach 0 (point B approaches point A).

A rule follows this theorem on page 214, stating that the equation of any line tangent to a given conic (c) at a given point (x_o,y_o) on the conic can be calculated by using the following substitutions:

- substitute $x\cdot x_o$ for x^2
- substitute $y\cdot y_o$ for y^2
- substitute $\frac{x\cdot y_0+y\cdot x_0}{2}$ for $x\cdot y$
- substitute $\frac{x+x_0}{2}$ for x
- substitute $\frac{y+y_0}{2}$ for y

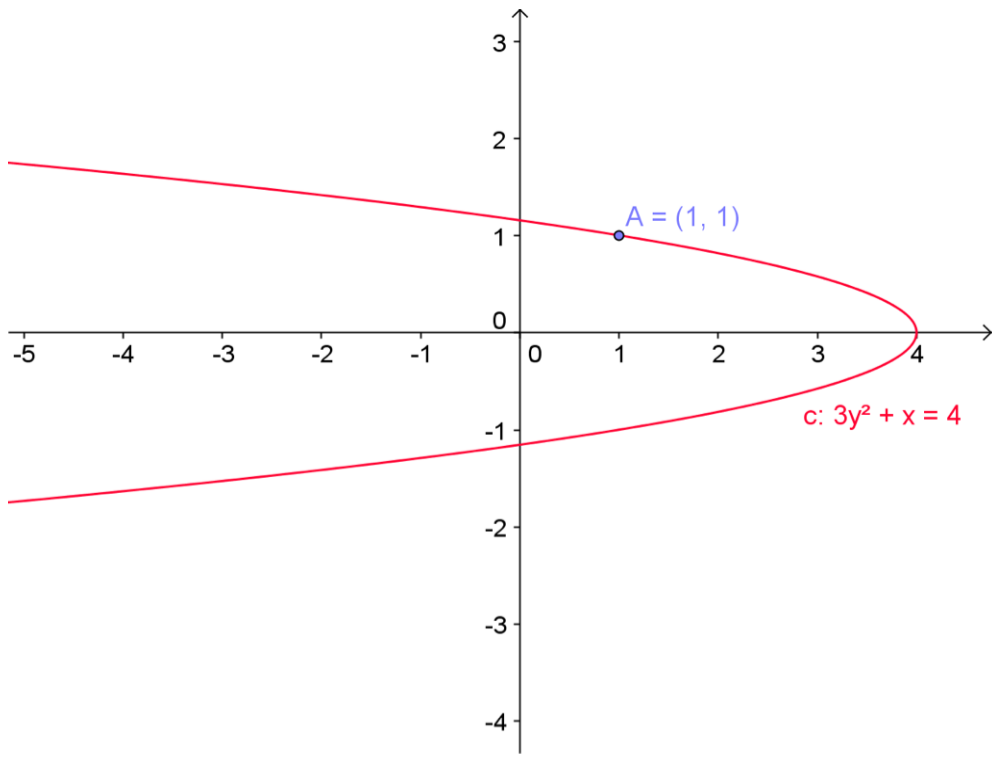
While the (x_o,y_o) represents the coordinates of the given point on the curve, the pair (x,y) from the final equation represents the coordinates of the locus point of the tangent curve.

Example 1:

We begin with an easy example, such as finding the equation of a tangent to a parabola $3y^2+x=4$, at the point (1,1) on the curve.

The graph of the parabola will look as shown in Diagram 1.

Diagram 1



Solution to Example 1:

Applying the rule shown in Method I, where y^2 becomes $y\cdot y_o$ and x becomes $\frac{x+x_0}{2}$, we find (after the required substitutions) that the equation of the tangent line is:

(t) $3\,y\cdot y_o + \frac{x+x_0}{2} = 4$

Since $x_o = 1$ and $y_o = 1$, we get:

(t) $3y + \frac{x+1}{2} = 4$

Let's multiply everything by 2 first:

(t) $2 \cdot 3y + 2 \cdot \frac{x+1}{2} = 2 \cdot 4$

Now we simplify:

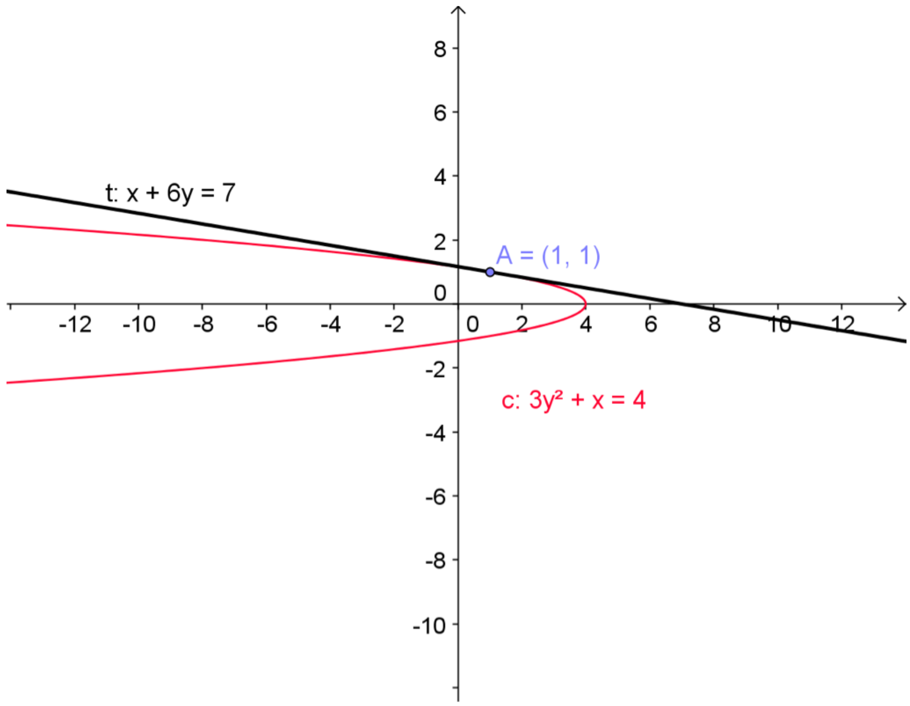
(t) $6y + x+1 = 8$

We need to subtract 1 from each side and this will lead us to a standard form of

(t) $x+6y=7$ for the tangent line

The graph of this tangent can be shown on the graph from Diagram 2.

Diagram 2



Example 2:

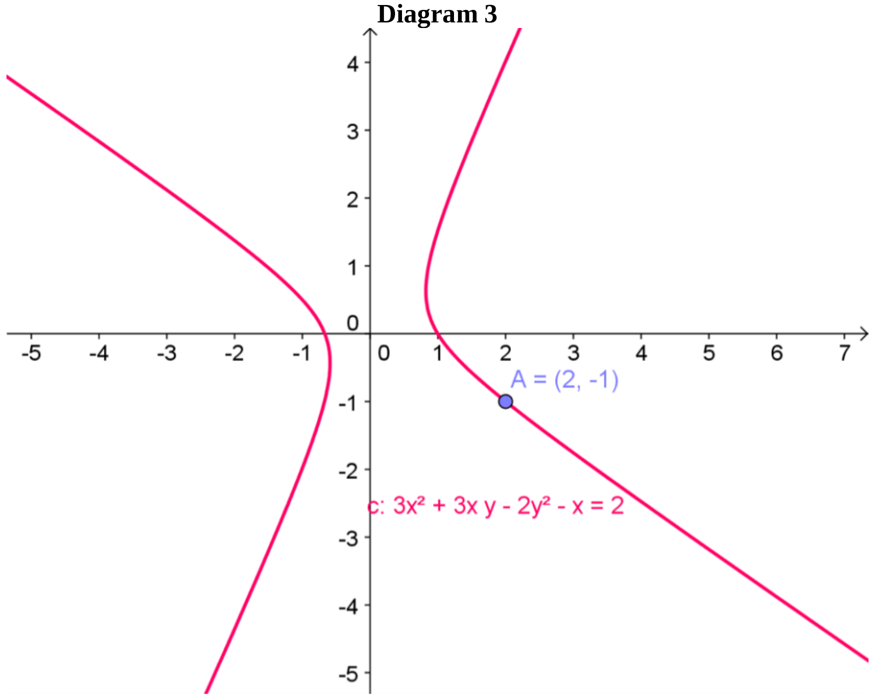
Let's try now with a more complex conic, such as the rotated hyperbola

(c) $3x^2 + 3x \cdot y - 2y^2 - x = 2$

We want the equation of the line tangent to the point (2,-1), which also lies on the curve.

The graph of this conic looks like the one shown in Diagram 3:

Diagram 3



Solution to Example 2:

Here we apply the same Method I and make the substitutions for the tangent line, with the respect to $x_o=2$ and $y_o= -1$. Hence,

- x^2 becomes $x \cdot x_o$, and since $x_o=2$, x^2 will be replaced by 2x
- y^2 becomes $y \cdot y_o$ and since $y_o= -1$, y^2 will be replaced by –y
- x·y becomes $\frac{x \cdot y_0 + y \cdot x_0}{2}$ and due to the substitutions of $x_o=2$ and $y_o= -1$, the fraction becomes $\frac{-x+2y}{2}$, and finally,
- x becomes $\frac{x+x_0}{2}$ that changes to $\frac{x+2}{2}$ due to the same $x_o=2$ substitution

So, the equation of the tangent is:

(t) $6x+\frac{3}{2}(-x+2y)+2y-\frac{1}{2}(x+2)=2$

Let's multiply everything by 2:

(t) $2 \cdot 6x+ 2 \cdot \frac{3}{2}(-x+2y)+ 2 \cdot 2y- 2 \cdot \frac{1}{2}(x+2)=2$

Now, perform the operations:

(t) $12x+3(-x+2y)+4y-(x+2)=4$

Apply the distributive property:

(t) $12x-3x+6y+4y-x-2=4$

Combine like terms:

(t) $8x+10y-2=4$

Add 2 both sides:

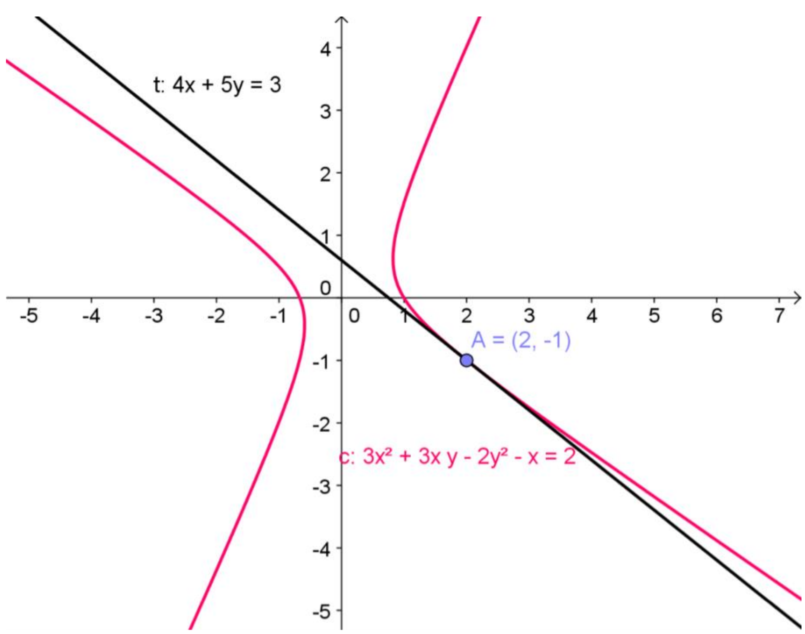
(t) $8x+10y=6$

And finally, divide everything by 2:

(t) $4x+5y=3$

The graph can be seen in Diagram 4.

Diagram 4



Method II: Finding the equation of a tangent line to a higher-degree curve:

In order to find the equation of the line tangent to a higher-degree curve, we use the same method, but this time in iteration. After each step, the degree of the curve is decreased by 1, 2, or more, yielding a lower-degree curve that shares the same tangent line at point (x_o, y_o). Consequently, a 3rd and a 4th degree curve will become a 2nd –degree one, a 5th and a 6th degree curve will become a 3rd one and so on.

For example,

- x^3 can be written as $x \cdot x^2$, and therefore can be substituted as $x \cdot x_o \frac{x+x_0}{2}$ since $x \cdot x_o$ replaces x^2 and $\frac{x+x_0}{2}$ replaces x
- x^4 can be written as $x^2 \cdot x^2$, and therefore can be substituted as $x^2 \cdot x_o^2$ since $x \cdot x_o$ replaces x^2 (twice)
- x^5 can be written as $x \cdot x^2 \cdot x^2$, and therefore can be substituted as $x^2 \cdot x_o^2 \frac{x+x_0}{2}$, again, since $x \cdot x_o$ replaces x^2 (twice again) and $\frac{x+x_0}{2}$ replaces x
- x^6 can be written as $x^2 \cdot x^2 \cdot x^2$, and therefore can be substituted as $x^3 \cdot x_o^3$

and so on. Of course, the same substitutions can be done for the y variable, as well.

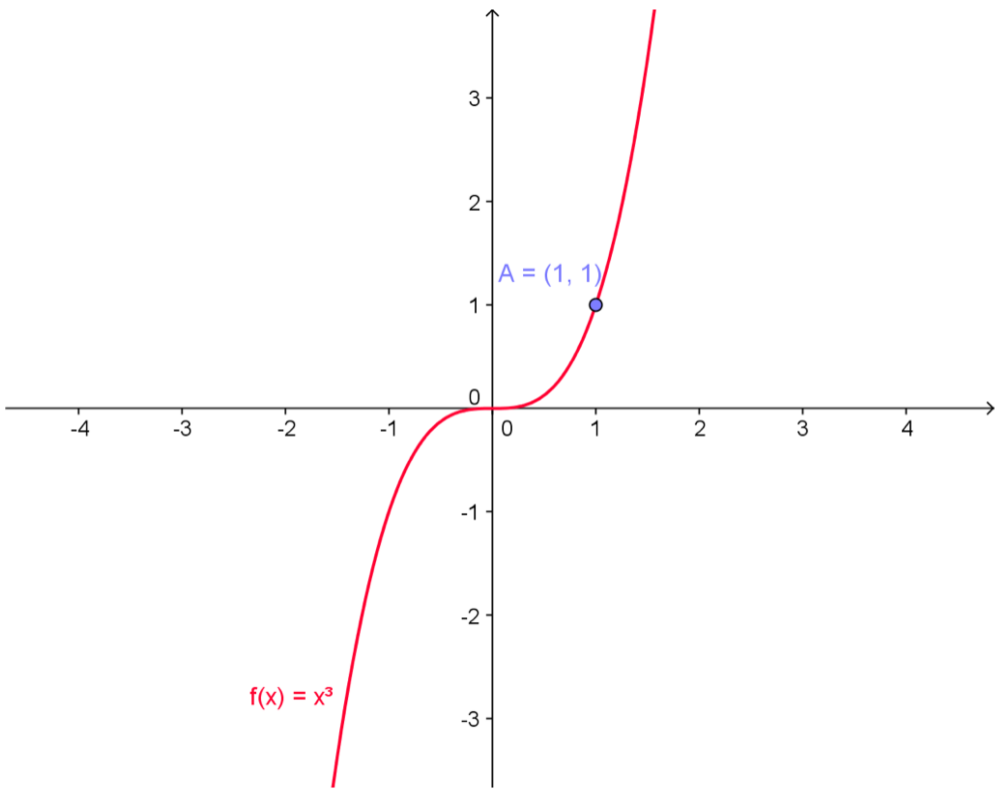
Example 3:

Let's find the tangent line to $y=x^3$ at the point $(x_o, y_o)= (1,1)$.

Solution to Example 3:

First, let's graph the curve and the point, as shown in Diagram 5.

Diagram 5



Applying the substitutions from Method I and Method II to the given curve:
(c) $y=x^3$ and with respect to $x_o = 1$ and $y_o= 1$,

- y becomes $\frac{y+y_0}{2}$, meaning $\frac{y+1}{2}$ since $y_o= 1$, and
- x^3 becomes $x \cdot x_o \frac{x+x_0}{2}$, meaning $x\frac{x+1}{2}$ since $x_o = 1$

Consequently, the equation of the curve (c) $y=x^3$ becomes

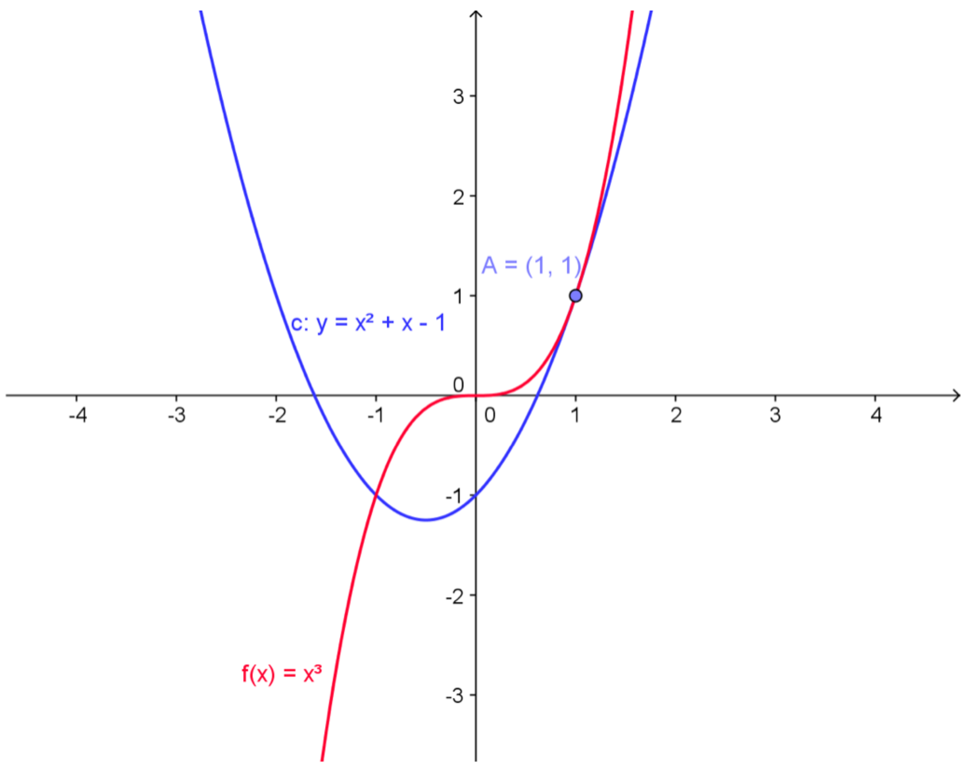
$\frac{y+1}{2} = x\frac{x+1}{2}$.
Multiplying all terms of the equation by 2, the following equation is produced:
(d) $y+1=x(x+1)$

We apply the distributive property:
(d) $y+1= x^2+x$

and after subtracting 1 from both sides, the equation simplifies to
(d) $y=x^2+x-1$

The graph of the resulting curve is shown in Diagram 6.

Diagram 6



Interestingly enough, if we continue this method of substitution, and we substitute all the variables from the parabola, we obtain:
(t) $y=3x-2$, that represents the tangent at point (1,1) for both curves.
Here are the 2nd round of substitutions made for the quadratic curve (d) $y=x^2+x-1$ that leads us to (t) $y=3x-2$:

- y is replaced by $\frac{y+y_0}{2}$ meaning $\frac{y+1}{2}$ since $y_o= 1$
- x^2 is replaced by $x \cdot x_o$, meaning $x\cdot 1$ since $x_o = 1$
- x is replaced by $\frac{x+x_0}{2}$ meaning $\frac{x+1}{2}$ since $x_o = 1$

With all these substitutions in place, the equation of the curve (d) $y=x^2+x-1$ changes to
(t) $\frac{y+1}{2}= x+ \frac{x+1}{2} \quad -1$

We multiply everything by 2:
(t) $2\cdot \frac{y+1}{2}=2\cdot x+ 2\cdot\frac{x+1}{2} \quad -1$

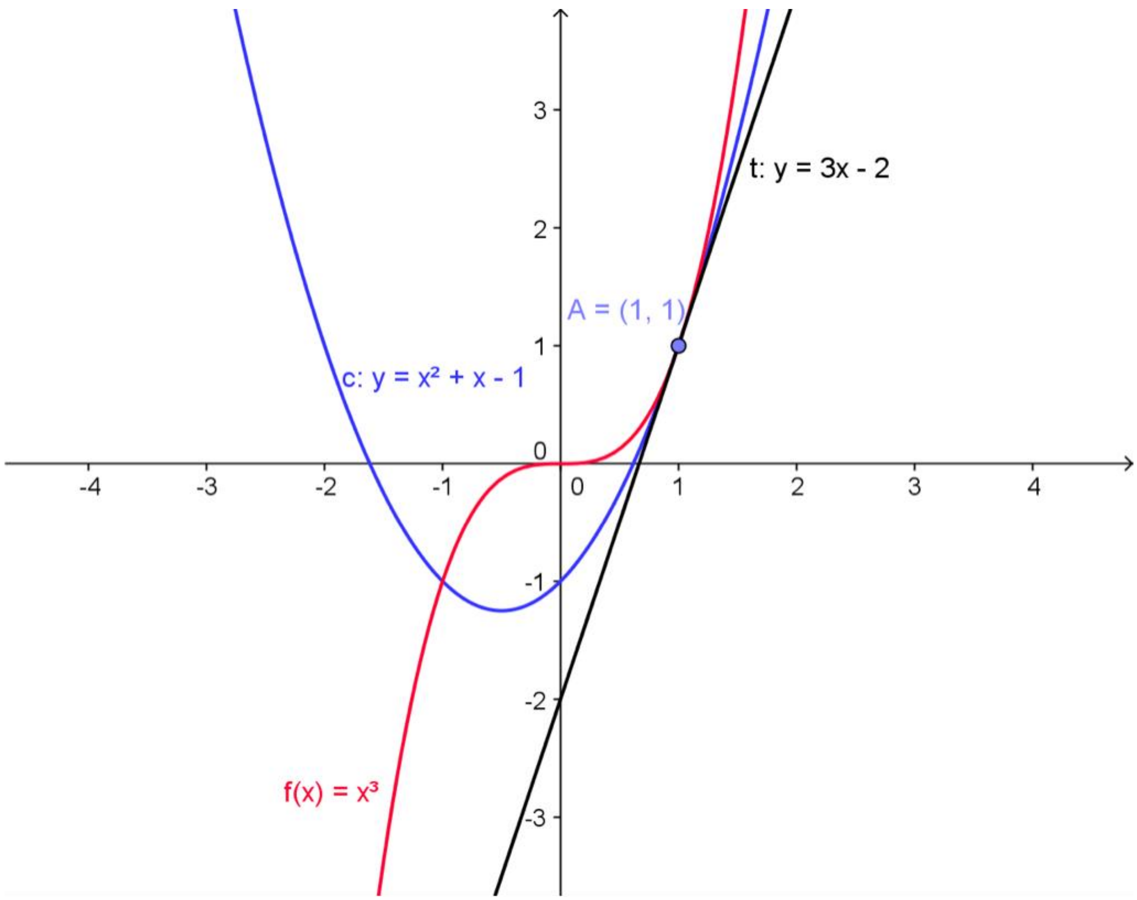
We now perform the operations:
(t) y+1=2x+x+1-2

We add like terms:
(t) y+1=3x-1

And finally, subtract 1 both sides:
(t) y=3x-2

The graph of this tangent (t), together with the original curve (c) and the quadratic (d) are shown in Diagram 7.

Diagram 7



Example 4:
Let's find the tangent equation at point (1,-5) to the curve $y= -x^3-5x^2+1$.

Solution to Example 4:
After the following substitutions (given in Method I and Method II):

- x^3 as $x\cdot x_o\frac{x+x_0}{2}$ with $x_o=1$ will change in $x\frac{x+1}{2}$
- x^2 as $x\cdot x_o$ again, with $x_o=1$ will change in x
- y as $\frac{y+y_0}{2}$, with $y_o= -5$ will change in $\frac{y-5}{2}$,

we reach the following equation:
(d) $\frac{y-5}{2}=-\frac{x(x+1)}{2}-5x+1$

Again, we multiply the entire equation by 2:
(d) $2\cdot\frac{y-5}{2}=2\cdot\frac{x(x+1)}{2}-2\cdot 5x+1$,

Perform the multiplications:
(d) $y-5=-x(x+1)-10x+2$

Apply distributive property:
(d) $y-5=-x^2-x-10x+2$

Add like terms:
(d) $y-5=-x^2-11x+2$

And finally, by adding 5 both sides, we obtain the quadratic equation:
(d) $y=-x^2-11x+7$

During the second round of substitutions, we switch again

- y with $\frac{y-5}{2}$ since y is replaced by $\frac{y+y_0}{2}$ with $y_o= -5$
- x^2 with x, since x^2 is substituted by $x\cdot x_o$ and $x_o=1$
- and x with $\frac{x+1}{2}$ after the substitution of $x_o=1$ in $\frac{x+x_0}{2}$

and we obtain the tangent line equation (t) $\frac{y-5}{2}=-x-\frac{11}{2}(x+1)+7$

Multiply again everything by 2:
(t) $2\cdot\frac{y-5}{2}=-2\cdot x-2\cdot\frac{11}{2}(x+1)+2\cdot 7$

Perform the operations:
(t) $y-5= -2x-11(x+1)+14$

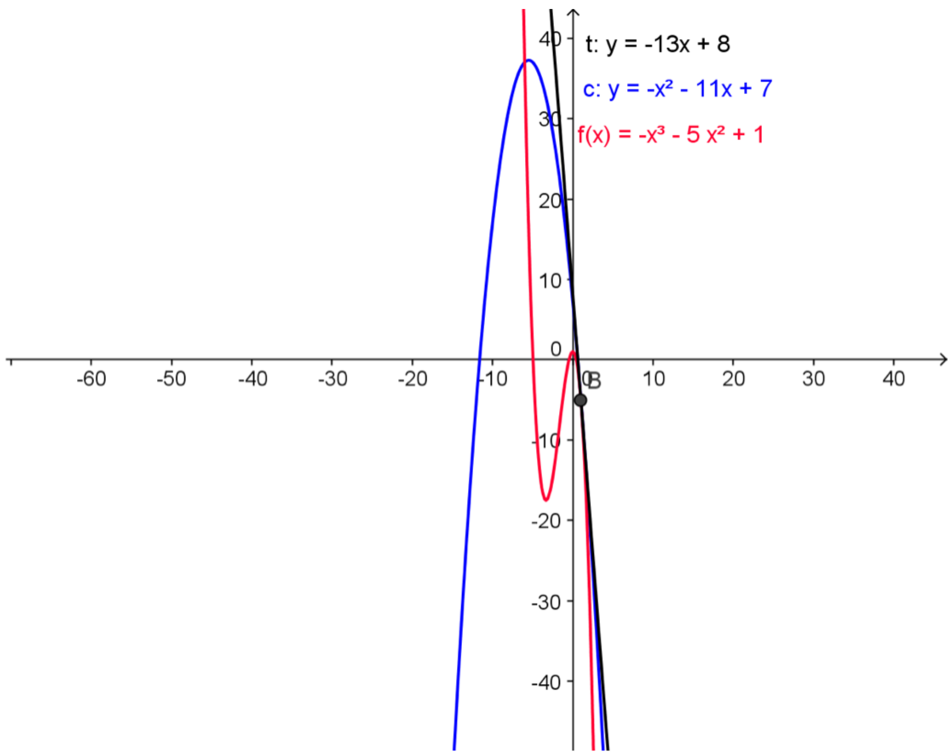
Apply distributive property:
(t) $y-5= -2x-11x-11+14$

Add like terms:
(t) $y-5= -13x+3$

Finally, add 5 both sides and the equation of the tangent line follows:
(t) $y= -13x+8$

The graph for all these curves and the tangent can be seen in Diagram 8.

Diagram 8



Example 5:

This time, let's try a 4-th degree curve, such as y= $-2x^4+3x^2+5$ and find its tangent line at the point (1, 6).

Solution to Example 5:

In order to find the solution to this problem, based on both Methods I and II, we have to re-write

- x^4 as $x^2 \cdot x^2$ and each of the x^2 will be replaced by $x \cdot x_o$ meaning $x \cdot 1$. Consequently, x^4 becomes x^2
- x^2 as $x \cdot x_o$ meaning $x \cdot 1$ and since $x_o = 1$
- y as $\frac{y+y_0}{2}$, meaning $\frac{y+6}{2}$ due to the fact that $y_o = 6$

These substitutions change the equation of the curve (c) y= $-2x^4+3x^2+5$ to the following quadratic:

(d) $\frac{y+6}{2} = -2x^2+3x+5$.

As usual, first we multiply everything by 2 in order to cancel the fraction:

(d) $2 \cdot \frac{y+6}{2} = -2 \cdot 2x^2+2 \cdot 3x+2 \cdot 5$

We continue with the multiplications:

(d) $y+6=-4x^2+6x+10$

Since there are no like terms, we subtract 6 both sides:

(d) $y=-4x^2+6x+4$.

The second round of substitutions performed on the (d) $y=-4x^2+6x+4$ curve and based again on Methods I and II, will produce the following changes:

- y changes as $\frac{y+y_0}{2}$, meaning $\frac{y+6}{2}$, because $y_o = 6$
- x^2 changes as $x \cdot x_o$ meaning $x \cdot 1$ since $x_o = 1$
- x is replaced by $\frac{x+x_0}{2}$, meaning $\frac{x+1}{2}$, since $x_o = 1$

All the above substitutions will give us the tangent line (t) $\frac{y+6}{2} = -4x+6\,\frac{x+1}{2}+4$.

We multiply everything by 2:
(t) $2\cdot \frac{y+6}{2} = -2\cdot 4x + 2\cdot 6\,\frac{x+1}{2} + 2\cdot 4$

We perform the multiplications:
(t) $y+6=-8x+6(x+1)+8$

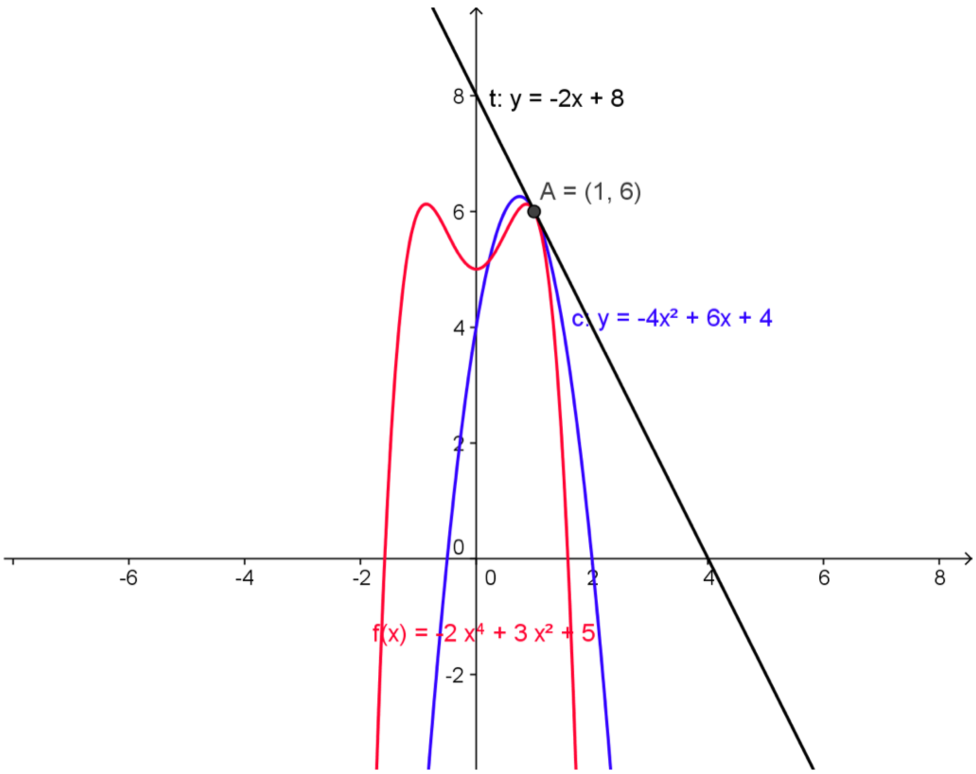
Next, we need to apply the distributive property:
(t) $y+6= -8x +6x+6+8$

We see some like terms that we need to address:
(t) $y+6= -2x+14$

And finally, we subtract 6 both sides, obtaining the tangent line:
(t) $y= -2x+8$

The graph is illustrated in Diagram 9.

Diagram 9



If we were to try rewriting x^4 as $x\cdot x^3$, the problem would force us to continue the factoring process until we obtain the power of 2, and therefore x^4 should become $x^2 \cdot x^2$. Generally speaking, all the higher degree terms of a polynomial need to be reduced to a lower degree by rewriting them as a product of a first or second power combination. Consequently, a 5th degree term will be rewritten as $x^2\cdot x^2\cdot x$, a 6th degree as $x^2\cdot x^2 \cdot x^2$ and so on.

As stated earlier in this article, the rigorous mathematical demonstration for the technique presented in Method I can be found on page 213 in Franklin's *The Elements of Analytic Geometry*, a book published in 1904, and which can be viewed online at the Google Books site. The method for a higher-degree curve is presented as a generalization or a repetition of the first method and since I didn't come across any proof, I am challenging the reader to bring one or a counter example.

I believe that this method offers a more primitive and somewhat easier alternative to its derivative twin, generating, as a bonus, equations of lower-degree curves having the same tangent at the given point. Moreover, this method can be used to find equations of lines tangent to a curve at a given exterior point, as well. Given its simplicity, I can see students without calculus skills being able to utilize this method to compute equations for tangents to curves of different degrees and make connections later on with derivatives. I sincerely hope that many math teachers will consider using these methods in their classrooms.

The graphic part of this presentation was done with the help of the GeoGebra program, a free software that can easily be downloaded from the www.geogebra.org site.

References

Bourne, M. Tangents *and Normals*, Interactive Mathematics, www.intmath.com

Franklin, P.,& Sullivan, A. (1904). *The Elements of Analytic Geometry*, Ginn & company, Harvard University, digitized on Google Books, p. 212-214. Direct link: http://books.google.com/books/reader?id=BKMLAAAAYAAJ&printsec=frontcover&output=reader

Osgood, W., & Graustein, W.(1921). *Plane and Solid Analytic Geometry*, The Macmillan Company, Wisconsin-Madison University, digitized on Google Books, p. 185

Tanner, J., & Allen, J. (1808) *An elementary course in analytic geometry*, American book company, Michigan University, digitized on Google Books, p. 206.

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Merging Process Standards and Inquiry: A Model for Mathematics Teachers

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Abstract

For years, mathematics teachers have been expected to incorporate the process standards of problem solving, communication, reasoning and proof, connections, and representation into their lessons as they help students to master important concepts. However, we have not had a widely accepted framework that shows teachers how this might be done. In this article, we provide a framework we have developed while working with and conducting research on meaningful inquiry-based instruction.

Problem solving has long been at the core of the reform movement in mathematics. NCTM (2000) has suggested that, in addition to problem solving, communication, mathematical reasoning, connections, and representations are also essential mathematical processes. These processes function both as “ends” and as “means.” In other words, teachers should help their students become proficient at these processes (e.g., good problem solvers) while students use them to learn and begin to own mathematical content.

Though these processes are fundamental to our standards, mathematics teachers have not had a widely accepted model for designing and implementing lessons that incorporate them in a manner that promotes strong conceptual understanding. Other disciplines have also called for strategies that better engage students to help them master content at deeper levels. Inquiry is a term widely accepted in many areas, and, through our work with dozens of teachers over the past four years, we have come to regard inquiry as an appropriate way to think about the merging of the five process standards into a cohesive structure.

However, though we have agreed upon the term, we recognize that inquiry has different meanings for different people (Anderson, 2002), so, while we advocate for inquiry-based instruction, we need to clarify what we mean by it. We have come to refer to content-embedded inquiry, a framework that unites the process standards with the content standards. Specifically, content-embedded inquiry is a strategy that employs the five process standards to help students master key content ideas identified in the Standards. See Figure 1.

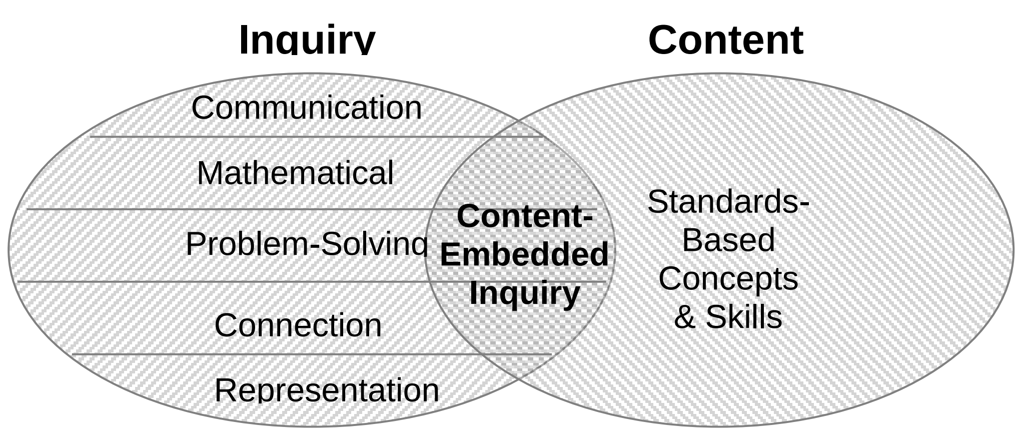

Figure 1: Content-Embedded Inquiry: The Intersection of Process & Content

Content-embedded inquiry is a change from traditional instruction, in which the teacher typically reviews homework, explains and demonstrates an algorithm for solving a particular type of problem, asks the students to attempt one or two problems on their own, and then provides an assignment for students to master and automate the algorithm. This classroom practice frequently leads to superficial understanding and promotes memorization over mathematical thinking (Brownell, 1954, 86; Schmidt, McNight, & Raizen, 2002). Content-embedded inquiry also differs from what we call activity mania in which teachers engage students in superficial activities that only scratch the surface (Schmidt, McNight, & Raizen, 2002). Our model for content-embedded inquiry is the 4E x 2 (read "4E by 2"), an instructional model we have developed and researched for more than three years.

The 4E x 2 Instructional Model

The 4E x 2 is designed to help teachers create and implement effective content-embedded inquiry. Our research has shown that the 4E x 2 can help teachers unite the process and content standards in a manner that increases student motivation and student achievement.

The 4E x 2 captures key components of inquiry as identified in the 5E model (Bybee et al., 2006), a model used widely in science education, by using Engage, Explore, Explain, and Extend as the key stages in the inquiry process. Combined with these are two additional components, formative assessment and reflective practice, which have been shown to improve student achievement. In our model, formative assessment refers to intentional, specific steps undertaken by the teacher at each of the stages of inquiry, and reflective practice refers to the teacher looking back over the instructional process and making informed decisions concerning the next steps to take. Figure 2 suggests the foundation for our Model.

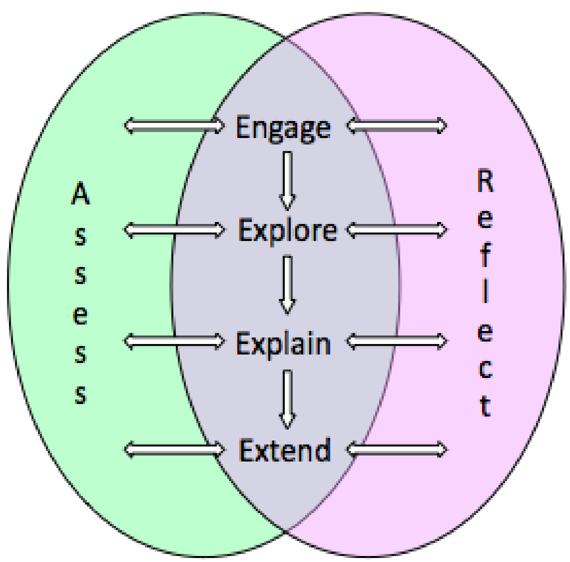

Figure 2: Overview of The 4E x 2 Instructional Model

In the Engage stage of instruction, teachers assess prior knowledge and identify misconceptions while generating student interest. During Explore, students actively investigate the mathematics. Guided by the teacher, students then analyze and interpret their findings in the Explain stage. The concepts developed are then generalized and applied to other contexts in the Extend stage.

Each of these four stages is linked directly to the process standards. First, Problem Solving is woven throughout all four Es. Content-embedded inquiry centers on a rich question, preferably one generated by the students, and the entire process revolves around solving the problem to drive toward the underlying mathematical concepts.

Mathematical Reasoning is most prominent during the Explore and Explain stages. During the Explore stage, students design a strategy for solving the problem, determining what information or data they need to collect, and then determining what they have found. In the Explain stage, students analyze and interpret their results, addressing not only what happened but why.

Communication, though evident at every stage, is most prominent during the Explain stage. Students should be held accountable for explaining to others what they have found and why they believe things work the way they do. This can be done orally, visually, in writing, or through a combination of these.

Representation is also perhaps most conspicuous in the Explain stage, although it is also key during the Explore and Extend stages. During the Explain stage, students must represent their ideas, using pictures, graphs, equations, and words to demonstrate their findings. Too often we insist upon representing problems, solutions, and concepts only algebraically; this is certainly a critical representation in many situations and often the goal of our instruction, but there are many times when, say, a graphical representation can be more powerful than an algebraic one. Connections should be one of the points of emphasis in the Engage and Extend stages. During the Engage stage, students should determine if the problem relates to others they have seen before and decide whether previous strategies they have used are likely to work. In Extend, we try to deepen students' understanding, connecting the concepts to other areas within mathematics and to other disciplines. Different contexts that embed the same underlying mathematical principles should come to light at this time.

All five processes are essential for effective content-embedded inquiry to occur, and the 4E x 2 provides a model for teachers to integrate all five into their teaching. It is important to note that the stages of inquiry are not fixed in order. For example, it may be appropriate to Engage, Explore1, Explain1, Explore2, Explain2, Extend1, Explore3, Explain3, Extend2. However, there are two immutable principles: 1) students should be engaged from the beginning with a meaningful problem, and 2) exploration must precede explanation of the concept, not vice versa as occurs in traditional mathematics instruction.

An Example of the 4E x 2

The initial step in lesson development is determining the standards to be addressed. The example provided here addresses an important characteristic of a measure of center. However, it can also be used to develop an intuitive understanding of absolute value, improve and reinforce students' mental arithmetic capabilities, and provide an opportunity to gather, organize and analyze data.

Engage

As NCTM states, teachers must "pose tasks that are based on sound and significant mathematics" and "engage students' intellect, develop students' mathematical understandings and skills, [and] stimulate students to make connections and develop a coherent framework for mathematical ideas" (NCTM, 1991, p. 25). We believe that the following problem, entitled "Where Should It Be?", meets these criteria.

> *A fast food chain sets up five restaurants along a highway at mile markers 2, 4, 16, 28, and 50. The owner of the chain needs a distribution center to service the restaurants. She will have five trucks, one for each of the restaurants. Of course, she wants to save money, so she wants the total number of miles the trucks will travel to be as small as possible. Where should she locate the distribution center?*

Engaging students entails not only motivating them but also identifying their prior knowledge and exposing their misconceptions. During the Engage stage, teachers seek to answer some or all of the following: Do students understand the question? What do they expect the answer to be and why? What misconceptions might they have? How might this problem connect to others they have studied? Engage provides an awakening of the mind so that learning can take place. When thinking is not perturbed, students are left unengaged and passive. On the other hand, when students' prior knowledge is activated, more significant learning is likely (Donovan & Bransford, 2005).

In the Engage phase, students begin to call upon appropriate strategies or identify a need to develop new ones. Connections are also important; as teachers evaluate prior knowledge, they identify whether students have made connections between the context of the problem and the mathematics they might need to solve it.

Assess. To determine the level of student understanding, teachers might ask the following: What information did students' predictions supply? Was prior knowledge sufficient to tackle the problem? Did they understand what was asked, and were they sufficiently interested to pursue a solution? Were their misconceptions to confront?

Reflect. As teachers reflect upon the Engage stage, they should decide whether the students are ready to proceed or need remediation. The key is for the teacher to reflect early on and make a purposeful decision as to the next step.

Explore

Exploring a problem is critical to inquiry-based learning. If this is among students' first true inquiry-based experiences, teachers may need to provide hints and suggestions to guide them; nevertheless, they should avoid directly telling the students how to explore the problem. For instance, students may ask if they should be concerned with the round trip distance or the one-way distance. This decision is best left to them; later on they can determine how their decision affects the solution. In so doing, this could lead to a discussion of the distributive property.

Students must be given sufficient time to explore. If, after several minutes, they cannot get started, then the teacher might suggest that they act it out or by perhaps set up a number line. Sadly, when we gave this problem to older students, including college students, we found that they immediately sought a formula or abstract solution. In fact, two doctoral students "solved" the problem within a couple of minutes, providing an incorrect "proof" of their answer. Had they been more willing to explore, they may have found their error and corrected their thinking.

Most students start with a somewhat random guess, determining the total distance from their guess to the five stores, and then try something different to see if they can do better. A question to help guide these students is whether they can develop a systematic method for keeping track of their guesses and the results they produce. As they explore, they are developing their mathematical reasoning; this puts more of the responsibility for learning on them.

Assess. Teachers should assess the amount of guidance students need during the exploration, being sure to assess everyone, not just the few students who may regularly speak out. If time allows, having students present their explorations can help other students and the teacher understand how others approached the problem and shed light on mathematical thinking. Requiring written descriptions of their exploration can also be valuable and can take less class time.

Reflect. What was the quality of the students' exploration? Were they able to gather and organize data? Could they justify their approach? How much scaffolding did students need, and what implications does this have for future explorations? Could students share their explorations in meaningful ways? Are students ready to move on, or do they need remediation?

Explain

Students should be actively involved in explaining their results and the processes used to obtain them. Additionally, students should also be able to justify their results, providing a convincing argument for their solution. This can take the form of a presentation, a written explanation (perhaps done individually), or a class discussion.

Teachers might ask how this problem connects to other contexts, topics, or patterns students have come across before. For example, this problem provides an opportunity to make connections to absolute value. For example, to determine the total distance from mile marker 20 to the five stores, calculating (50-20) + (28-20) + (16-20) + (4-20) + (2-20) does not work; the absolute value of each difference is needed before summing.

We have also used this problem to strengthen students' intuition about integers by comparing the total distance if the distribution is moved from, say, mile marker 20 to mile marker 19. In this case, the distance decreases by one mile to the stores at mile markers 2, 4, and 16 and increases by one mile to the stores at mile markers 28 and 50, for a net decrease of 1 mile. At this and the subsequent Extend stage, content-embedded inquiry can provide a myriad of opportunities to make connections.

Ultimately, we want students to master the underlying mathematical ideas. To achieve this end, students need to be involved in the mathematical discourse. This does not relegate teachers to roles of observers, but it does mean that they need to listen to and understand students' thought processes *before* providing a summative or synthesizing explanation (Bransford et al., 2000).

As students formulate an explanation, the communication standard is addressed. Additionally, students may use a variety of representations (tables, graphs, verbal descriptions) to express their findings. As students present their findings, connections among groups' various representations can be made. Through rich mathematical discourse, students are challenged to justify their conclusions and analyze the results of other groups, employing the reasoning and proof standard.

Assess. Teachers should ask questions such as: How well did students explain their process and results? Do they understand? How much scaffolding was needed? Can students be pushed a little further, or are they struggling? This assessment can occur through presentations, individual student writing, class discussions, lab reports, or other means. What is important is that the teacher assesses the explanations, listening carefully in order to make appropriate instructional decisions.

Reflect. Did the students have strong evidence for their claims? Were they able to solve the problem? Could they determine the quality of their work? Do they need to work on their explanations or go back and explore more? Most importantly, could they present and defend a valid solution?

Extend

The Extend stage provides an opportunity to solidify and deepen student understanding and thinking. At this stage, knowledge is transferred or generalized to new situations. For this problem, numerous extensions are appropriate:

- How does your result relate to measures of center? Will this happen with any 5 stores?
- What would happen if a 6th store is added at mile marker 100?
- Will this always happen when there are 6 stores? Why? 7? 8? *n* stores?
- What implications does this have for measures of center? Why?

As discussed, other extensions are possible as well. Ideally, students will generate additional questions that they wish to pursue. Depending on the goals of the lesson, the time available, and the needs and interests of the students, the Extend stage can lead to additional Explore and Explain stages. This is also a time to emphasize connections both to other mathematical concepts and to ideas in other disciplines.

Assess. Teachers can assess formatively through discussions, student presentations, lab reports, or other forms. How competent were the students in making generalizations? Can they transfer their knowledge to new situations? Can they provide meaningful proofs of their conclusions?

Reflect. Teachers should think back upon the extensions. Do students need to engage and explore again? Is it fair to provide a summative assessment on the mathematical ideas?

Summary and Conclusions

Compare the effects of the above problem with a typical math lesson in which the teacher tells students that the median minimizes the total distance to the values in any data set. Not only will students gain a deeper appreciation for the median while addressing all five of the process standards, they will internalize the ideas and reinforce other mathematical ideas. Content-embedded inquiry, as framed by the 4E x 2, provides teachers a strategy for integrating the process standards into instruction. Consequently, the processes serve dual purposes. First, they lead students to deeper understanding of mathematical ideas. Second, the experience that students gain improves their facility with the processes themselves; for example, by solving challenging and meaningful problems, students become better problem solvers.

In our studies with both pre-service and in-service teachers, we have found that the 4E x 2 provides a sound framework that improves practice by merging the content and process standards. With guidance and by working in teams, teachers can develop, pilot, and refine meaningful lessons that deepen student learning. For an online template that guides the creation of lessons and to view lessons that other teachers have created using the 4E x 2 Model, visit www.clemson.edu/iim/lessonplans.

Instead of simply providing activities for students, the 4E x 2 allows teachers to see content-embedded inquiry as a strategy that improves their instruction. No longer will teachers struggle to answer, “When will I ever use this?” The content is embedded in the inquiry (Marshall, Horton, & Smart, 2009).

References

Anderson, R. D. (2002). Reforming science teaching: What research says about inquiry. *Journal of Science Teacher Education*, *13*(1), 1-12.

Brownell, W.A. (1954/1986). The revolution in arithmetic. Arithmetic Teacher, October, '86, 38-42.

Bybee, R. W., Taylor, J. A., Gardner, A., Scotter, P. V., Powell, J. C., Westbrook, A., et al. (2006). *The BSCS 5E instructional model: Origins, effectiveness, and applications*. Colorado Springs: BSCS.

Donovan, S. M., & Bransford, J. D. (2005). *How students learn: history, mathematics, and science in the classroom*. Washington, DC: National Academies Press.

Marshall, J. C., Horton, B., & Smart, J. (2009). 4E x 2 Instructional Model: Uniting three learning constructs to improve praxis in science and mathematics classrooms. Journal of Science Teacher Education.

National Council of Teachers of Mathematics (2000). *Principles and Standards for School Mathematics*. Reston, VA.

National Council of Teachers of Mathematics (1991). *Professional Standards for Teaching Mathematics*. Reston, VA.

Schmidt, W. H., McNight, C. C., & Raizen, S. A. (2002). A splintered vision: An investigation of U.S. science and mathematics education. from http://imc.lisd.k12.mi.us/MSC1/Timms.html

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