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THE MATHMATE

The Official Journal of the
South Carolina Council of Teachers of Mathematics

Mission Statement: The mission of THE MATHMATE is to feature articles about innovative mathematical classroom practices, important and timely educational issues, pedagogical methods, theoretical findings, significant mathematical ideas, and hands-on classroom activities and disseminate this information to students, educators and administrators.

THE MATHMATE, the official journal of the South Carolina Council of Teachers of Mathematics, is published online three times each year – January, May, and September.

Submission Requirements: Submissions for THE MATHMATE should be no more than 15 pages in length not counting cover page, abstract, references, tables, and figures. Submissions of more than 15 pages will be reviewed at the discretion of the editorial board. Submissions should conform to the style specified in the *Publications Manual of the American Psychological Association* (6th ed.). All submissions are to be emailed to scmathmate@gmail.com as attachments with a completed Submission Coversheet as page 1 and the article starting on page 2. Click here to download THE MATHMATE Submission Coversheet.

Submitted files must be saved as MSWord, RTF, or PDF files. Pictures and diagrams must be saved as separate files and appropriately labeled according to APA style. Copyright information will be sent once an article is reviewed but authors should not submit the same article to another publication while it is in review for THE MATHMATE.

Submission Deadlines: Submissions received by October 1 will be considered for the January issue, February 1 for the May issue, and June 1 for the September issue.

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The MathMate

Table of Contents

Message from the SCCTM President

Dear Members,

Welcome to the second issue of THE MATHMATE for 2014! We are excited to be able to get THE MATHMATE published and sent to our members twice. We hope you will enjoy these articles and continue your willingness to contribute. We are also extremely grateful to our editor, Gina Dunn, who has worked very hard to publish THE MATHMATE for us.

I hope that you appreciate the time and effort that the writers, reviewers, and editorial board have spent in publishing this journal. Please feel free to use the information you gain from the articles in your classroom. You may copy the activities from the journal and use them in your classrooms and share them with your colleagues. We are already working on the next issue and you play a vital role in the success of THE MATHMATE. Please take the time to submit an article showcasing either a classroom activity or research project that would be beneficial to other mathematics educators. Information about submission of articles can be found in THE MATHMATE and on our website (http://www.scctmprogram.org) or you can send it directly to our editor, Gina Dunn, at SCMathMate@gmail.com.

We are in the planning stages of this year's SCCTM conference. The dates are November 6-7 in Myrtle Beach, South Carolina. Share your teaching ideas and practices by presenting a session or holding a workshop. At our annual fall conference, we have a variety of presentation formats and invite educators to submit Speaker Proposals for consideration. The deadline has been extended to June 5, and the form can be found at http://www.scctmprogram.org/SCCTM-Conference.html. Our goal is to promote mathematics in our state and positively impact student achievement in mathematics.

The SCCTM has increased its membership rates for the first time since 2006. Please renew your membership and encourage your colleagues to join the SCCTM to take advantage of the current rates before the increase goes into effect on July 1st. You can go to www.scctm.org and follow the links under About Membership.

As the school year ends, we hope you have a fun, safe, and relaxing summer!

Sincerely,
Jennifer E. Wilson
SCCTM President

Announcements

Upcoming Deadlines:

Outstanding Contributions to Mathematics Education Award and Richard W. Riley Award nominations are due **July 15, 2014**.

Pre-service Scholarship and Educator's Scholarship applications are due **September 15, 2014.**

Upcoming Conference Information and Deadlines:

SCCTM 2014 Annual Fall Conference, Myrtle Beach, South Carolina,
November 6 – 7
Speaker Proposal Form extended to **June 5, 2014**

Membership News:

SCCTM dues will increase effective **July 1, 2014.** Renew/Rejoin before the increase takes place by completing the form on page 7. New members are also welcome to take advantage of this opportunity.

Renew your NCTM membership online and designate *South Carolina Council of Teachers of Mathematics* for the affiliate rebate.

If you would like your announcement to appear in the next issue of THE MATHMATE, please email all information to SCMathMate@gmail.com by September 1, 2014. Announcements will be published at the discretion of THE MATHMATE Editorial Board.

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This form becomes obsolete after June 30, 2014 and can not be honored after that date!

IMF 2013.02

Guest Editorial

The tools may have changed, but the best way to grow as an educator is still listening to the voice of experience!

Susan Martin Peeples

As a former editor of THE MATHMATE, there was “no” way I could say no when asked to write an article. When I was editor, ‘cut and paste’ was literal. Even my 95-year old mother-in-law remembers helping put the labels on the newsletters and then sorting them by zip code all over the den floor. Change is frequently good! And I saw a lot of changes in technology in the 40 years I was in the classroom.

I started teaching in 1968 and my first technological advancement came the next year when I got a box of colored chalk. Think about how that helped demonstrate steps in algebra or diagrams in geometry. Even with the latest tricks-of-the trade, color in teaching helps many different learning styles. I always insisted that students have at least 3 different colors with them. Girls often like more colors, but even the guys will use red and blue pens. But I *teach* them how to use the colors for learning, not just to decorate.

Especially as a young teacher, my most appreciated new piece of equipment was the overhead projector. Unlike with the blackboard, I could explain and demonstrate while looking at my students. I used the overhead projector almost every day until my last day in the classroom in 2008. And once again, the vis-à-vis pens allowed me to use color and were much easier to clean than the yellow grease pencils. Another advantage of the overhead was that I could prepare drawings and slides ahead of time and not use valuable class time writing while students sat bored and getting in trouble. Getting the most out of class time was always a priority.

The SmartBoard came in toward the end of my formal career. WOW!! Because I could open the on-line text and write right on the problems, I could use my prep time for creating more activities for my students. Of course, the SmartBoard has a plethora of other applications also. Most days, I used the white board, the overhead AND the SmartBoard during one lesson. I was all over the place and that kept the students involved. I must admit I am not sure how I would teach all the essential skills using a tablet, but I know that I would figure out how.

During my 40 years in public education, The South Carolina Teachers of Mathematics and National Council of Teachers of Mathematics were essential to my growth as a teacher. These venues kept me current and prevented tunnel vision. This is where I learned the ‘good stuff’ like Escher and Fibonacci before they were in textbooks. I learned so many tips from other teachers.

Watching other teachers teach is one of the greatest benefits I got from several years of working in curriculum and supervision. But, to be honest, I had known from pre-school that I was going to teach. I just did not know what or whom. So, I have watched teachers all my life, evaluating and filing away what I thought effective and what I did not like or think fair. I continue to learn from students I tutor. I glean a lot from the assignments, pace, tests and notes. Go watch another teacher every chance you get and invite others into your classroom.

When I had student teachers, there were several things I stressed. You have to love your students and you have to *know* your content. We all have ‘the class from hell’ pretty regularly, but you have to believe in your students in order to communicate that you want them to succeed and you expect excellence. You expect so much that you will work harder than they to help them. We are the professionals and that is our task. Our job is not to be voted most popular, but to be the best instructor. Sometimes the best teachers are not appreciated until the next year. *Knowing* the content means knowing how what you are teaching now is used in future courses: the difference between what is essential and what is 'nice to know.' We have always had standards that influence what we teach, but do you really know what is in the next course?

A final passing thought: mastery of Algebra I is essential for almost all our students, regardless of future career/education plans. Most of the students I tutor in geometry through pre-calculus have problems because they cannot do the basic Algebra I manipulations and understand the essentials. Repeating Algebra I should be a good option and not one fought by parents and guidance counselors because of scheduling problems or stigma of repeating a course. When students have a strong foundation in Algebra I, they move through future courses more confidently and frequently find that they really like math.

Even though I am no longer in the classroom, I am still passionate about teaching mathematics and hope you are, too.

Mathematical Modeling of a Function

Amber Simpson, Jennifer Blethen, and Stefani Mokalled
Clemson University

Abstract

This article presents a mathematical modeling problem in a real-world context. The problem provides students with data on the temperature of cooling water and guides them through experimenting with different parameters to gain a better understanding of their significance. Using Excel, students are led to determine the best-fit function by minimizing the sum of the squared errors.

What do a swimming pool, a cup of coffee, and a bowl of soup all have in common? These objects are all affected by the temperature of their surroundings. More specifically, these objects follow Newton's Law of Heating and Cooling: the change in temperature of an object is proportional to the change in the ambient temperature. The goal of this activity is to have students use Microsoft Excel, software not commonly known or used by students, in order to find the best-fit function by experimenting with different parameters and minimizing the sum of the squared errors.

It is important to note that there is a variety of technologies that could be used to model an exponential function. But when attempting to model this exponential relationship with graphing calculators, we found that they did not have the capabilities of determining the limit of an exponential function in terms of a real-world context. Hence, we used Microsoft Excel as it is commonly accessible to most high school students. While there are numerous technologies available in today's society, it is crucial to remember that "[t]echnology should not be used as a replacement for basic understanding and intuitions; rather, it can and should be used to foster those understandings and intuitions" (NCTM, 2000, p.25).

This activity enhances student learning by allowing them to experiment with different parameters before discovering the real-world exponential model. Several Common Core State Standards for Mathematics (CCSSM) are incorporated in this activity including Building Functions (F-BF), Linear, Quadratic, and Exponential Models (F-LE), and Interpreting Categorical and Quantitative Data (S-ID). For example, CCSSM standard F-BF.1.b states that students will "combine standard function types using arithmetic operations. For example, build a function that models the temperature of a cooling body by adding a constant function to a decaying exponential, and relate these functions to the model." (CCSSO 2010, p. 70).

In addition, this activity is aligned with several of CCSSM (2010) practice standards: (1) make sense of problems and persevere in solving them; (2) construct viable arguments and critique the reasoning of others; and (3) model with mathematics. Students will be challenged to make conjectures within the context of the problem, as well as monitor and evaluate their progress. Making predictions and evaluating conclusions help develop students' reasoning skills in mathematics, which may lead to greater student learning (e.g., Boaler & Staple, 2008). Students will be asked to communicate their answers and justify their decisions, promoting rich classroom discourse. In addition, the activity is based on a real-world problem, in which students interpret results within the context of the problem, as well as make connections between various representations and concepts promoting the notion that mathematics is more than a static and discrete body of knowledge (Ma, 2010).

Excel Setup encourage

As part of this activity, an analysis tool, the Solver, is used to find the optimal value of a target cell. In this case, the Solver adjusts the parameters of the mathematical model as it finds the minimum value of the sum of squared errors. The tool can be found under the Data tab. If the Solver tool is not currently accessible, it is easy and free to add into Excel. Under the File menu, open up the Options menu and click on Add-Ins. Near the bottom of the box, there is a drop-down menu entitled Manage. Select Excel Add-In and then click on the Go button. From here, select the option Solver Add-In by simply clicking in the box on the left and click Okay. Different versions of Excel may have slightly different procedures.

Next, set up a template in Excel to include the proper headings (see Figure 1).

	A	B	C	D	E	F	G	H
1	Time (mins.)	Temp (F)	Predicted	Error	Error Squared		y = A*B^x + C	
2	0	152.9					A	
3	3	147.7					B	
4	6	144.2					C	
5	9	139.2						
6	12	134.4						
7	15	130.1						
8	18	125.4						
9	21	121.8						
10	24	118.6						
11	27	115.4						
12	30	112.5						
13	33	109.9						
14	36	107.7						
15	39	105.6						
16	42	103.6						
17	45	101.7						
18	48	99.9						
19	51	98.3						
20	54	96.7						
21	57	95.5						
22	60	94.2						
23	63	93.0						
24	66	91.9						
25	69	90.8						
26	72	89.7						
27	75	88.8						
28	78	87.9						
29	81	87.0						
30	84	86.2						
31	87	85.5						
32	90	84.7						
33								
34								

Figure 1. Data points from our experiment. Illustration of how to set-up Excel.

To become more familiar with working in Excel, we have provided the necessary steps below to set up a template for this particular problem. To set up the template, begin by entering the headings in Row 1 (refer to Figure 1). Then place the mouse between any two columns, say column A and column B, and double click. Excel automatically adjusts the column width based on the length of the text. Continue this process to adjust all the columns. To accommodate teachers who wish to use this activity, but do not have the time to collect data points on their own or as part of the class activity itself, we have provided data points from our experiment as well, yet we encourage students to collect their own data. As an alternative to using hot water, students can take the temperature of cold water over a span of an hour or an hour and a half every three minutes. Though to create the model we describe here, make sure any ice has melted before beginning the data collection. It might also be interesting to compare models using a variety of containers such as a coffee mug, a thermos, or a “to-go” cup. Furthermore, consider asking students to compare a liquid, such as sweat tea, with water, and discuss how additives may affect how fast or slowly a liquid warms up or cools down. However, to become familiar with the activity and with Excel, we suggest practicing with our data prior to having students collect their own data. In addition, we encourage anticipating students’ responses to the questions posed throughout the activity to help facilitate the discussion and make connections among students’ responses and to other mathematical concepts (Smith & Stein, 2011).

Excel Activity

First, with the template open, create a scatter plot of the data points, first highlight Column A and Column B, the independent and dependent variables. Under the Insert Tab at the top of the page, select the Scatter arrow to pull up several options for the scatter plot. Select the “Scatter with only Markers.” While the scatter plot is selected, discuss with students an appropriate chart title, axes titles, and horizontal and vertical scale intervals. These changes and many more are found under the Chart Tools Tab. For example, to label the horizontal axis, select the graph. Under the Layout Tab, select Axis Titles, Primary Horizontal Axis Title, followed by Title Below Axis. Figure 2 displays the scatter plot of the data with the titles we included. Ask students if the scatter plot represents growth or decay and have them defend their answers within the context of the data. What will the graph look like over the next hour and why? How cool will the water be after ten or twenty hours and why?

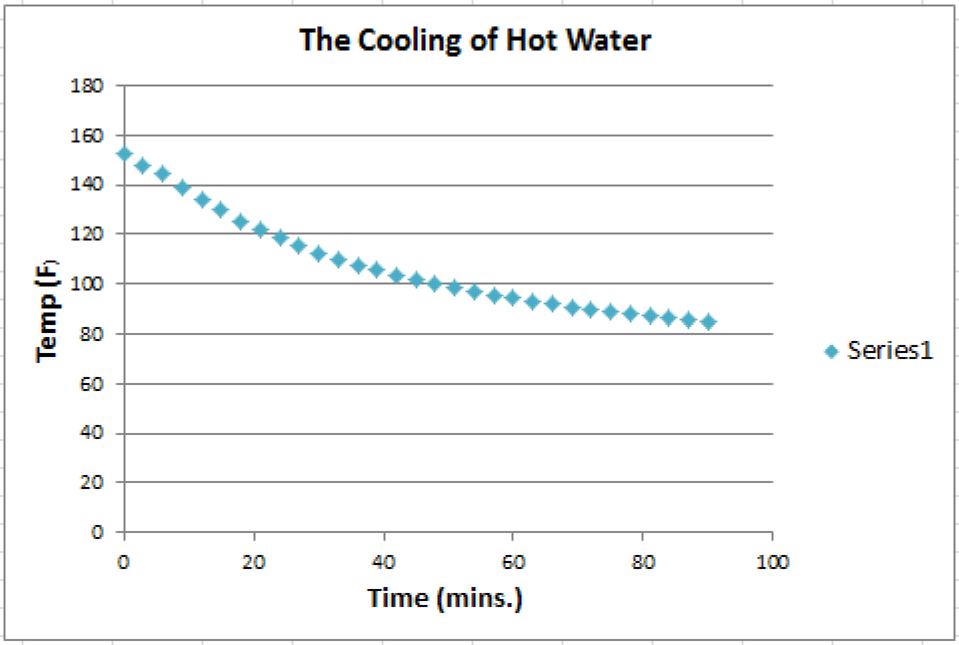

Figure 2. Scatter plot of the data.

Our model for the temperature of the water as it cools toward room temperature is $y = A*B^x + C$, as Newton's Law of Heating and Cooling suggests. In our model, x represents the time that has elapsed, measured in minutes, and y the temperature of the liquid, which we measured in degrees Fahrenheit. Our model is similar to an exponential decay model, except that instead of a horizontal asymptote at $y = 0$, we have a horizontal asymptote at $y = C$, which represents room temperature. In other words, a typical exponential decay function has been shifted vertically C degrees. For example, suppose that the exponential model for our data is $y = 101.00(.98)^x + 74.64$. The value of 74.64 indicates that the room temperature was approximately 74.64 degrees Fahrenheit, and this is the temperature that the water will, in time, approach.

Parameter A represents the difference in the initial temperature of the water from the room temperature. Note that in a typical exponential model, A represents the initial value. For our model, A represents the initial displacement above room temperature. Returning to the example above, $y = 101.00(.98)^x + 74.64$, the initial temperature of the water (after it was heated and set out in the room) was 101 degrees above room temperature, or 175.64 degrees Fahrenheit. This value is found by adding 101.00 + 76.46.

Parameter B represents the percent of difference from the current temperature of the water to the room temperature that it retains each minute. A B-value greater than 1 would represent that the difference between the liquid's temperature and room temperature is increasing, an impossible scenario for a liquid just sitting out in the room. Consequently, B must be less than 1, representing that the difference between the liquid temperature and room temperature is shrinking. We know from the example that, on average, the water retained 98% of the difference between its current temperature and room temperature each minute. Another way of thinking about B is that the difference between the temperature of the water and the room temperature decreased, on average, by 2% every minute.

Have each student substitute possible values for parameters A, B, and C in cells H2, H3, and H4, respectively, and encourage students to talk about their values and explain why these values were chosen in regards to the problem.

Next, calculate the predicted values by typing =H2*(H3^A2) + H4 into Cell C2. This substitutes the independent value, time in minutes, into the model and calculates the temperature that the model predicts. The $ symbol in the formula keeps the values within those cells static regardless of where the equation is copied and pasted. Instead of having to retype the Excel formula into each cell, place the mouse on the bottom-right corner of Cell C2 and drag the mouse down the column to cell C32. The remaining predicted values are then calculated. Answers vary at this point depending on the students' values for parameters A, B, and C. Encourage students to play around with the numbers of A, B, and C and describe what is happening to the predicted values in Column C.

At this point, we want to add the data points of the predicted values just calculated in Column C onto the scatter plot. Right click on one of the points on the graph. Then choose Select Data, click on Add, and type in Predicted for the Name. Next, click on the empty box to the right of Series X, select Cell A2, and drag the mouse through to Cell A32. Likewise, click on the box under Series Y and select Cells C2 through C32. Then select the okay button twice. See Figure 3 as an example. How does changing parameters A, B, and/or C change the Predicted graph and/or the numerical values in the Excel worksheet? What is the goal of changing the parameters?

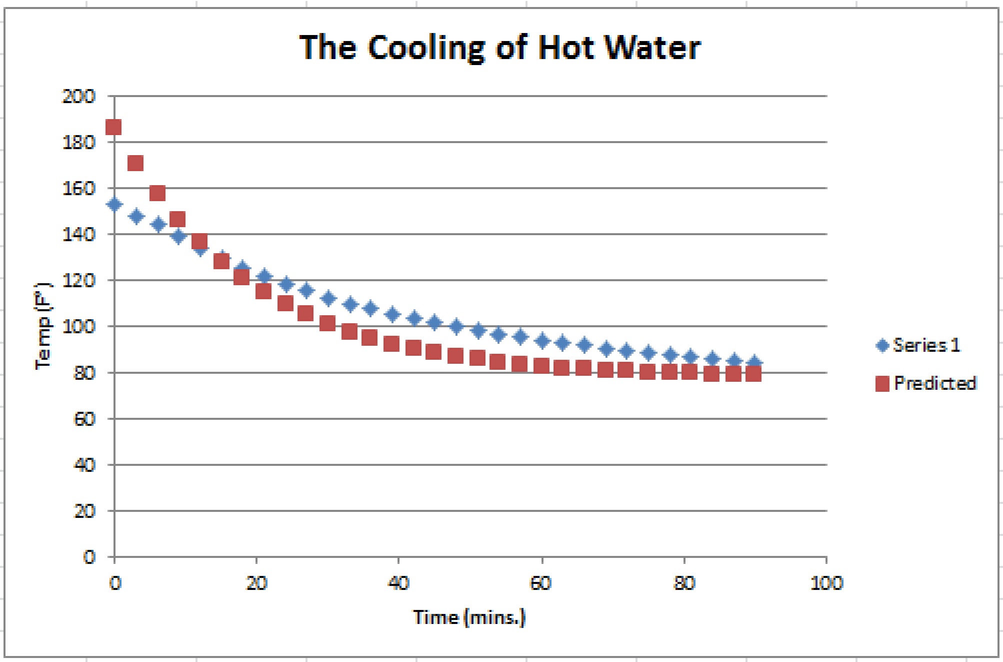

Figure 3. Graphical representation of predicted values.

The errors, or residuals, in the model are the differences between the actual y-values, or the temperature in this case, and the predicted y-values from the regression equation. Therefore, subtract the predicted value from the actual value, or subtract the values in Column C from the values in Column B. In cell D2, type =B2-C2, followed by enter. Again, place the mouse on the bottom-right corner of the Cell D2 and drag the mouse down the column to Cell D32. Next, have students find the sum of the errors by adding all the values in Column D. Type =SUM(D2:D32) in Cell D34. We suggest typing the text SUM in Cell C34 as a means to label this value. Challenge students to find values for the parameters to minimize the value of the sum of the errors. Furthermore, students should consider what the sum of the error means in relation to the graph. For instance, what does it mean if the sum of the error is equal to zero?

At this point, students should develop an understanding of why minimizing the errors is not sufficient for a "best-fit" model because it is possible to minimize the errors, but not produce a model with any true meaning; namely the sum of the positive and negative errors equal zero. The accepted practice in statistics is to square the errors, and then attempt to minimize the total of these squared errors. To minimize the sum of squared errors, square the values in Column D or the predicted error. In Cell E2, type =D2^2, followed by enter. Again, place the mouse on the bottom-right corner of the cell and drag the mouse down the column to Cell E32. To find the sum of the squared errors, select Cell D34, and with the mouse on the bottom right-hand corner, drag it over to Cell E34.

At this point, once again, encourage students to change the values of the parameters in the regression equation to find the least sum of squared errors. Have a conversation about how changing the values of A, B, and C individually affects the value of the sum of the squared error, as well as the other values (and graph) within the spreadsheet. Consider having a friendly competition among the students to determine who can adjust A, B, and C and achieve the smallest sum of squared errors.

We're now ready to find the best-fit model. Open the Solver located under the Data tab. For set objective, type in E34 or select Cell E34; and then below, select Min. This tells Excel to find the minimum of the value located in cell E34, in this case, the sum of the squared errors. In the box entitled "By Changing Variable Cells," enter H2, H3, H4. Deselect "Make Unconstrained Variable Non-Negative" by clicking on the checkmark and GRG Nonlinear should appear on the drop down box for "Select a Solving Method" (see Figure 4). Click the Solve button. Another text box will appear; select Okay to keep the solver solution.

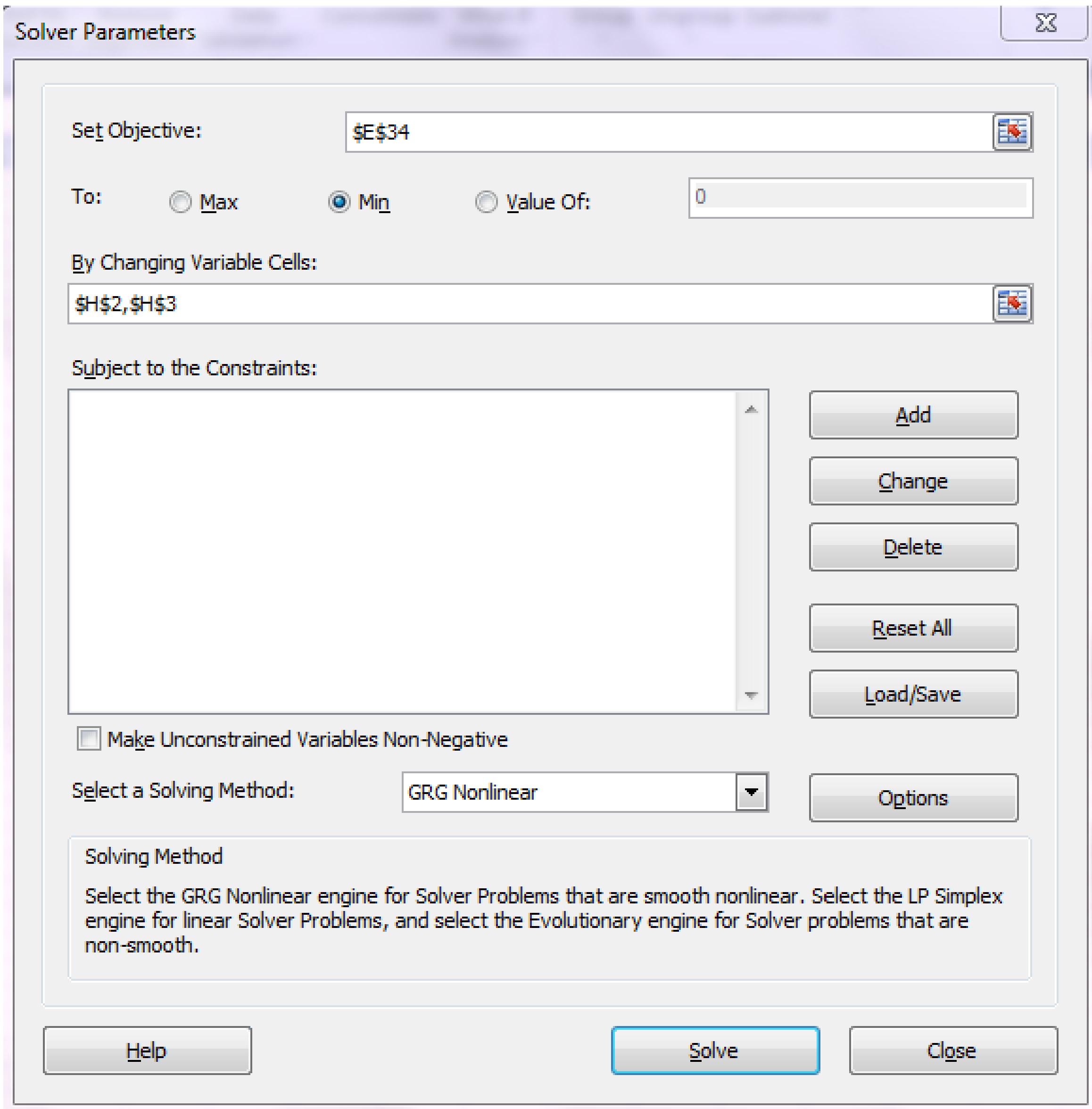

Figure 4. Image of solver tool.

From this, we can see that the model that represents our data is $y = 76.533(0.9749)^x + 77.33$ with a sum of squared errors of approximately 5.98. In our function $y = A*B^x + C$, the C value of 77.33 indicates that the room temperature was approximately 77.33 degrees Fahrenheit. The initial value of the temperature was 76.533 degrees Fahrenheit above room temperature, or 153.86 degrees Fahrenheit, which is calculated by adding 76.53 + 77.33. Furthermore, the value in Parameter B tells us that on average the water retained about 97.5% of the difference between its current temperature and room temperature every minute. Ask students what these values mean within the context of the problem, and how the values of A, B, and C differ from their initial guesses. How would the parameters of the function change if the initial temperature of the water were 174 degrees Fahrenheit? What would happen to the graph and why do you think this is so?

Conclusion

After completing this activity, students should have a better understanding of how to interpret the parameters of a mathematical function and what is meant by the "least squares" method. Using Excel, students employ critical thinking skills that allow them to explore the parameters involved in a model that is closely related to an exponential function. Students are challenged to justify their thinking and support their conjectures throughout the activity, promoting conversation within the mathematics classroom. The numerous real-world applications of this problem motivate students' interest and engage them in the activity. When asked about using Excel in mathematics class, a 9th grade student responded that "it would be helpful, but we do not use it that much…It could be useful if we're making charts and stuff." Thus, using this activity may demonstrate to students the benefits of tools such as Excel, not just for creating charts, but also for exploring mathematics while engaged in real-world problems.

In addition, consider applying this concept to other situations of exponential growth or decay such as changes in population, Olympic records, or radioactive half-life. In fact, this technique is appropriate for any type of mathematical model. Furthermore, we encourage working alongside a teacher of science, history, music, or any other discipline in order help students create connections between the different disciplines (NCTM, 2000).

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An Innovative Algorithm to Determine the Square of Integers with Only Repeated Digits and Integers of Two and Three Digits

Md Roshid Ahmad
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Abstract

Squaring numbers is an important mathematical skill that is often used in textual problems and in everyday life. In Bangladesh, calculators are not allowed through the U.S. equivalent of 7th grade, yet squaring numbers is often required. However, there is no satisfactory discussion on simplifying this process or finding an alternate algorithm that can generate interest while lessening the burden on students so that they can focus on the current lesson. Over the years I've sought shortcuts for students that would help them square numbers while they simultaneously practiced their multiplication. When I've introduced them to the strategies presented in this article, they become very attentive. With these strategies, after just a few minutes of instruction and practice, students are able to find the square of any integer with the same digit, the square of any two-digit integer, and the square of any three-digit integer. Afterwards, students can go directly to the result without doing any scratch work.

Squaring numbers is an important mathematical skill that is often used in textual problems and in everyday life. In Bangladesh, calculators are not allowed through the U.S. equivalent of 7th grade, yet squaring numbers is often required. However, there is no satisfactory discussion on simplifying this process or finding an alternate algorithm that can generate interest while lessening the burden on students so that they can focus on the current lesson.

In Bangladesh, multiplication is introduced in Class One (equivalent to 1st grade) and is emphasized frequently through Class Five. Over the years I've sought shortcuts for students that would help them square numbers while they simultaneously practiced their multiplication. In the secondary school, students are allowed to use calculators to perform arithmetic and are rarely interested in doing calculations by hand. However, when I've introduced them to the strategies presented in this article, they become very attentive. With these strategies, after just a few minutes of instruction and practice, students are able to find the square of any integer with the same digit (e.g., 6666), the square of any two-digit integer, and the square of any three-digit integer. Afterwards, students can go directly to the result without doing any scratch work! We challenge you to try these strategies yourself and then challenge your students, assuming they are in fifth grade or higher. One caveat: students in Bangladesh are expected to know their multiplication tables through 20, not just through 10 or 12.

In all cases, the position of the digit is considered from right to left. Thus the 1st digit refers to the 1's place, the 2nd digit refers to the 10's place, the 3rd digit refers to the 100's place, and so on.

The Square of an Integer Containing Only The Same Digits

An Example

Let's consider the square of 6666, or, equivalently, 6666^2. Our number, 6666, has 4 digits. First we square our repeated digit. In this case, $6^2 = 36$. We then multiply this number (36) by 1, 2, 3, 4, 3, 2, 1, 0, in that order. We stop at 4 and return because there are 4 digits in our number. Had there been 5 digits, we would have multiplied by 1, 2, 3, 4, 5, 4, 3, 2, 1, 0. Recall that the 1st digit refers to the 1's place, the 2nd digit to the 10's place, and so on. Here goes:

$36 \times 1 = 36$ (the 1st digit will be <u>**6**</u> and we'll carry the 3 to the 2nd digit)
$36 \times 2 = 72$ ($72 + 3 = 75$, the 2nd digit will be <u>**5**</u> and we'll carry the 7 to the 3rd digit)
$36 \times 3 = 108$ ($108 + 7 = 115$, so our 3rd digit is <u>**5**</u> and we'll carry the 11 to the 4th digit)
$36 \times 4 = 144$ ($144 + 11 = 155$, so our 4th digit is <u>**5**</u> and we'll carry the 15 to the 5th digit)
$36 \times 3 = 108$ ($108 + 15 = 123$, so our 5th is <u>**3**</u> and we'll carry the 12 to the 6th digit)
$36 \times 2 = 72$ ($72 + 12 = 84$, so our 6th digit is <u>**4**</u> and we'll carry the 8 to the 7th digit)
$36 \times 1 = 36$ ($36 + 8 = 44$, so our 7th digit is <u>**4**</u> and we'll carry the 4 to the 8th digit)
$36 \times 0 = 0$ ($0 + 4 =$ <u>**4**</u>, so our 8th and final digit is 4).

Thus our result, which you can find by reading up, is 44,435,556. Sure it seems tricky the first or second time, but by the third or fourth time, our students found they could do this both quickly and accurately. They also had fun with it.

In summary, the repeated digit is squared. This result is then multiplied by 1, 2, 3, ... (up to the number of digits the integer contains) and then by …..3, 2, 1, 0. (Do not repeat the multiplication with the number of digits.) All “carries” or “regroupings” move to the next digit. The results of the multiplication are written respectively as the 1^{st}, 2^{nd}, 3^{rd}, ... digits of the square.

Why Does This Work?
A 2-Digit Integer

Suppose a two-digit number has ‘x’ in every digit. The value of the number is $x\times10 + x$
Let, $x \times x = x^2$ = M. We will multiply M by 1, 2, 1 and 0, in that order.

Now, for the described process:
The first digit of the square = M×1.
The second digit of the square = M×2.
The third digit of the square = M×1.
The fourth digit of the square = M×0.

So, noting the values of each place value, we have $(x \times10 + x)^2$ = M×0×1000 + M×1×100 + M×2×10 + M. Simplifying, we have 100M + 20M + M. Recall that M = x^2, so is this the result we want?

$$
\begin{aligned}
&100x^2 + 20x^2 + x^2\\
&= x^2(100 + 20 +1)\\
&= x^2(121)\\
&= x^2(11)^2\\
&= x^2(10 + 1)^2\\
&= (10x + x)^2, \text{ which is what we were trying to find.}
\end{aligned}
$$

A 3-Digit Number

Suppose a three-digit number has ‘x’ in every digit. So the number is $x\times100 + x\times10 + x$.
Again suppose $x \times x = x^2$ = M. This time we will multiply M by 1, 2, 3, 2, 1 and 0. Will this give us the result we desire?

Now, the first digit of the square = M×1
The second digit of the square = M×2
The third digit of the square = M×3
The fourth digit of the square = M×2
The fifth digit of the square = M×1
The sixth digit of the square = M×0

So, we claim that $(x\times100 + x\times10 + x)^2$ = M×0×100000 +M×1×10000 +M×2×1000+M×3×100 +M×2×10 +M. Let’s check algebraically.

$$
\begin{aligned}
&10000x^2 + 2000x^2 +300x^2 + 20x^2 + x^2\\
&= x^2 (10000 + 2000 + 300 + 20 +1)\\
&= x^2(12321)\\
&= x^2(111)^2\\
&= x^2(100 + 10 + 1)^2\\
&= (100x + 10x + x)^2
\end{aligned}
$$

A 4-Digit Number

Suppose a four-digit number has ‘x’ in every digit. So, the number is $x\times1000 + x\times100 + x\times10 + x$. Again, let $x\times x = x^2 =$ M). We will multiply M by 1, 2, 3, 4, 3, 2, 1 and 0.

The first digit of the square = M×1
The second digit of the square = M×2
The third digit of the square = M×3
The fourth digit of the square = M×4
The fifth digit of the square = M×3
The sixth digit of the square = M×2
The seventh digit of the square = M×1
The eighth digit of the square = M×0

So, the square of (x×1000 + x×100 + x×10 + x)
= M×0×10000000 + M×1×1000000 + M×2×100000 + M×3×10000 + M×4×1000 + M×3×100 + M×2×10 + M

$$1000000x^2 + 200000x^2 +30000x^2 +4000x^2 +300x^2 +20x^2 + x^2$$
$$= x^2(1000000 + 200000 + 30000 + 4000 +300 + 20 +1)$$
$$= x^2(1234321)$$
$$= x^2(1111)^2$$
$$= x^2(1000 + 100 + 10 + 1)^2$$
$$= (1000x +100x + 10x + x)^2$$

The process is similar for larger numbers with repeated digits.

One More Example with Explanation

Suppose we want to find the square of 5555. First we square the repeated digit and label it M. So 5×5 = 25 = M. The integer has 4 digits. So we will multiply M by 1, 2, 3, 4, 3, 2, 1 and 0, respectively, and the results will be written in the place of 1st, 2nd, and 3rd ... digit of the square. Keep in mind that any “carries” will be added to the subsequent product.

1st digit of the square: M×1 = 25×1 = 25.	The 1st digit is 5
2nd digit of the square: M× 2 + R1= 25×2 + 2 = 52	The 2nd digit is 2
3rd digit of the square: M×3 + R2 = 25×3 + 5 = 80	The 3rd digit is 0
4th digit of the square: M×4 + R3 = 25 × 4 + 8 = 108	The 4th digit is 8
5th digit of the square: M×3 + R4 = 25×3 + 10 = 85	The 5th digit is 5
6th digit of the square: M×2 + R5 = 25×2 + 8 = 58	The 6th digit is 8
7th digit of the square: M×1 + R6 = 25×1 + 5 = 30	The 7th digit is 0
8th digit of the square: M×0 + R7 = 25×0 + 3 = 3	The 8th digit is 3

Remembering that the first digit is the 1’s place, we can now right the square of 5555:

5555^2 = 30,858,025

The Square of a Two-Digit Number

Process

To find the square of any 2-digit number,

- Square the 1st digit (remember that this is the 1’s place). This will be the 1st digit of the square.
- Add the 1st digit of the original number with the original number. Multiply this sum by the 2nd digit of the original number. The results are written as the 2nd, 3rd, and 4th digits of the square.

Examples with Explanation

Suppose we want to square 97.

- The 1st digit is 7. We square it to obtain 49. The 1st digit of the square is 9. We’ll carry the 4.
- Add 7 (the 1st digit of the original number) with the original number. This gives us 104. We multiply 104 by 9 (the 2nd digit of the original number), obtaining 936. We add the 4 that we carried, to get 940. These represent the other digits of the square.

We have found that $97^2 = 9409$.

Let’s try another. Suppose we want to square 54.

- Squaring the 4, we obtain 16. The first digit of the square will be 6 and we’ll carry the 1.
- Adding 4 with 54 (first digit of the original number and the original number) us 4 + 54 = 58. We then multiply 58 by 5 (the 2nd digit of the original number) and carry the 1. We get $58 \times 5 + 1 = 291$.

The square of 54 is 2916.

Why Does This Work?

Let’s call the number’s 1st digit a, and let’s call its 2nd digit b.
So, the number is = $b \times 10 + a$.
The sum of the number and its first digit = $b \times 10 + a + a = b \times 10 + 2a$
Let the 1st digit of the sum = x, the 2nd digit of the sum = y, and the 3rd digit (if any) of the sum = z.
Combining the two lines above, we have $b \times 10 + 2a = z \times 100 + y \times 10 + x$)
By our procedure,
The first digit of the square = $a \times a$
The second digit of the square = $b \times x$
The third digit of the square = $b \times y$
The fourth digit of the square = $b \times z$
Let’s look at the algebra to verify that process gives us the square of the original number:

$$\begin{aligned}\text{The square of } (b\times 10 + a) &= b\times z\times 1000 + b\times y\times 100 + b\times x\times 10 + a\times a\\ &= 10b(z\times 100 + y\times 10 + x) + a^2\\ &= 10b(b\times 10 + 2a) + a^2\\ &= 100b^2 + 2\times 10b\times a + a^2\\ &= (10b + a)^2\end{aligned}$$

The Square of a Three Digit Number

Process

Let an integer’s 1st digit = a, 2nd digit = b and 3rd digit = c
So, the number is = $c \times 100 + b \times 10 + a$
Hence, the first digit of the square = $a \times a$
The second digit of the square = $a \times b \times 2$
The third digit of the square = $a \times c \times 2 + b \times b$
The fourth digit of the square = $b \times c \times 2$
The fifth digit of the square = $c \times c$
The sixth digit of the square = the carries, if any, of the last result.

Why Does This Work?

$$\begin{aligned}\text{The square of } (c\times 100 + b\times 10 + a) &= c\times c\times 10000 + b\times c\times 2\times 1000 + (a\times c\times 2+b\times b)\times 100\\ &\quad + a\times b\times 2\times 10 + a\times a\\ &=10000c^2 + 2000bc + 200ac +100b^2 +20ab +a^2\\ &= (100c)^2 + (10b)^2 + a^2 + 2\times 100c\times 10b + 2\times 10b\times a + 2\times 100c\times a\\ &= (100c +10b + a)^2\end{aligned}$$

Example with Explanation

Suppose we want to find the square of 654.
The first digit of the number (a) = 4
The second digit of the number (b) = 5
The third digit of the number (c) = 6

$a\times a = 4\times 4 = 16$	The 1st digit of the square is **6**. We carry 1.
$a\times b\times 2 + R_1 = 4\times 5\times 2 + 1 = 41$	The 2nd digit of the square is **1**. We carry 4.
$a\times c\times 2 + b\times b + R_2 = 4\times 6\times 2 + 5\times 5 + 4 = 77$	The 3rd digit of the square is **7**. We carry 7.
$b\times c\times 2 + R_3 = 5\times 6\times 2 + 7 = 67$	The 4th digit of the square is **7**. We carry 6.
$c\times c + R_4 = 6\times 6 + 6 = 42$	The 5th digit of the square is **2**. We carry 4.
$R_5 = 4$	The 6th digit of the square is **4**. We’re finished!

Reading up, we find the square of 654 = 654^2 = 427716.

Conclusion

The aim of my work is to have the students able to do the calculation in their minds and write the square directly without doing any scratch work. To be efficient, students need to practice to reinforce the skills; also the steps for squaring a 3-digit integer are tricky and take some effort to memorize. From my experience, I have found that ***every*** student can do so, and I have also found that students can learn the strategies and be successful with them in just a few minutes. Just as important, I have found that some students who claim to dislike mathematics are intrigued by these strategies and, after being introduced to them, take a greater interest in the subject.

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Strategies for Integrating Interactive Whiteboards in Early Childhood Mathematics Centers

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Abstract

This article provides examples of how to incorporate Interactive Whiteboards (IWBs) into mathematics centers for early childhood classrooms.

District leaders, administrators, and teachers in public school systems across the United States invest a staggering amount of money into educational technologies; one estimate for technology-related expenditures in 2011 was $16 billion (McCrummen, 2010). Despite this investment, there is general agreement that, by and large, technology is not always being effectively integrated into PreK-12 instruction (Richtel, 2011). Many teachers use technology but often for Internet-based information retrieval (Greenhow, Robelia, & Hughes, 2009), presentational purposes (Smith, Rudd, & Coghlan, 2008), and communication and administrative tasks such as emailing and collecting or organizing data (Wozney, Venkatesh, & Abrami, 2006). While these common uses can be productive, they are not necessarily examples of effective integration.

In its technology position statement, the National Association for the Education of Young Children (NAEYC) defined effective uses of technology to be active, hands-on, engaging, and empowering (2012, p.6). Technology experiences should be shared between adults and children; support creativity, be explorative, and promote cognition and social interaction. These types of interactions with technology coincide with best practices in early childhood mathematics education. Early childhood teachers should encourage young children's emerging understanding of mathematics through engaging tasks that promote discourse and enable children to make connections between mathematical content and their everyday experiences. Ideally, integrating technology and mathematics in an effective manner can help young children make important mathematical connections. Using Interactive Whiteboards (IWBs) can be a vehicle for this integration. IWBs are electronic whiteboards that can be connected to a computer and a projector to display and manipulate content on a screen (Author 1, 2012). Devices like SmartBoards or Promethean Boards fall under IWB technology.

When used appropriately, IWBs can be interactive and engaging, and provide a social environment in which children can construct "their own understandings of the world and broaden their knowledge bases" (Johnson, 2010, p. 6). The reality of how they are being used, however, falls short of their potential. For example, many teachers are using IWBs as tools for explicit instruction and whole-class teaching (Author, 2012). Moreover, teacher-student dialogue and collaboration is minimal. Activities often do not demand higher order thinking and IWB activities are considered by children to be more passive than playful (Morgan, 2010). Although these limitations are present in the literature, it is possible for teachers to incorporate IWBs into the early childhood mathematics classroom in a manner that enhances mathematical tasks. This article provides two examples of how teachers can use effective mathematics practices while integrating the IWB through mathematics centers. Each example is grounded in the mathematical content taught during the early years of elementary school (PreK-grade 3) as defined by NAEYC (2012) and the National Council of Teachers of Mathematics (NCTM, 2000).

Characteristics of Effective Integration

Implementing strong mathematical tasks in the early childhood classroom is critical for building a foundational understanding of content and, perhaps more importantly, positive dispositions towards mathematics (NCTM, 2000; NRC, 2001). Mathematical tasks in the early childhood classroom should be developed around five characteristics (NCTM, 2000):

1. *Building Communities*. Children should be developing relationships with each other by working in a safe environment where all are encouraged to succeed; they should work together to investigate

mathematical tasks. Teachers should be cognizant of children's social and emotional needs when deciding on partners or groups.

2. *Increasing Discourse*. Children should feel confident enough to express their thoughts and opinions about the task. Teachers should encourage children to communicate by asking thought-provoking, open-ended questions and by structuring tasks, so that all children have a role to play.
3. *Using Materials*. Children should have the opportunity to explore a variety of materials and manipulatives that they can use to enhance a task. To facilitate these explorations, the teacher should choose materials that encourage mathematical thinking, meaning that playing with the material should not be the sole purpose of the lesson, but rather act as a means of getting to the mathematics.
4. *Making Connections*. Teachers should design tasks in which students can make connections between mathematics content and their everyday experiences to make mathematics meaningful. When mathematics tasks are relevant, children can bridge the gap between what they already know and what they are expected to learn.
5. *Focusing on Processes*. Tasks should encourage the development of conceptual understanding of mathematics content. Children should be problem solving, reasoning, and representing their understanding in a variety of ways.

Integrating an IWB in a mathematics task naturally falls under the "Using Materials" category listed above. However, a task is only effective if all of these characteristics are present. Therefore, using an IWB to present a mathematical problem to the class may be a good use of materials, but without encouraging discourse and collaboration through an investigation of the problem, the IWB can become a limitation rather than an innovative tool. Teachers need to develop strategies for implementing strong mathematical tasks while making use of the technology present in their classrooms.

An efficient way to integrate IWBs and mathematics in early childhood settings without compromising these characteristics of strong mathematical tasks is through the use of classroom centers. Centers provide opportunities for students to work in small groups on tasks while the teacher is either present, acting as a facilitator asking guiding questions, or not present, and students have to complete the task without assistance. While centers are often limited to the kindergarten level, their use in first through third grade (and above) can be very helpful in encouraging independent and reflective mathematical thinkers (Van de Walle, Karp, Lovin, & Bay-Williams, 2013). In addition, centers can be helpful when differentiating instruction to meet the needs of all students in the classroom. Tasks can be designed with multiple entry points (Van de Walle, et al., 2013) so that any child in the class can complete the work successfully. For example, a center task that encourages building number sense might require a pair of first grade students to decompose a set of apples into two baskets in a variety of ways. In this scenario, one pair might only be able to decompose a set of four apples while another might work with seven. An alternative strategy would call for all students to work with seven apples, but allow pairs to collaborate to identify multiple ways to decompose the set. While one student may only see seven as a set of three and a set of four, other students may provide additional ideas. What follows are two hypothetical examples of mathematics centers built around the IWB. The geometry example is geared towards second or third grade content while the algebra example focuses on patterning in kindergarten. In each of these examples, the IWB acts as a tool that enhances a strong mathematical task. Connections following each of these examples explain how the activities met the characteristics of strong mathematical tasks in early childhood settings.

Using the IWB in geometry

This center focuses on two-dimensional shapes and their relative positions in space. It was developed midway through a unit about identifying and describing attributes of two dimensional shapes. Ms. Guider, a second grade teacher (all names are pseudonyms) decided this center would be appropriate because students were able to identify the names of shapes and to describe some of their attributes, but were not yet able to see how shapes could be combined to make different shapes. To address these concepts, Ms. Guider makes use of tangrams, or puzzles consisting of the same seven shapes that are used specifically to form other shapes (often seen put together in a square or rectangle). To implement this center, Ms. Guider combines the use of manipulatives with technology and group collaboration. Students are expected to work in pairs to solve a tangram puzzle (four students in the center at a time). The puzzle is presented to them on the IWB as an outline of a shape; students are responsible for manipulating the virtual tangrams by rotating, reflecting, and translating pieces to make them fit the puzzle (see Figure 1 below). Students are given instructions on how to complete the center through a voice recording. This recording, narrated by the teacher, can be played by touching a picture of a microphone on the IWB. A voice recording, whether it is recorded by the teacher or a student, can be an engaging way to present instructions, provide feedback, or tell stories. Voice recordings are easy to create and attach to images within both the Smartboard environment (by selecting the image and choosing *Sound* from the Insert menu) and the Promethean environment (by selecting the image and choosing *Sound Recorder* from Tools→More Tools menu option). More detailed instructions on how to do this can be gleaned from a quick *YouTube* search.

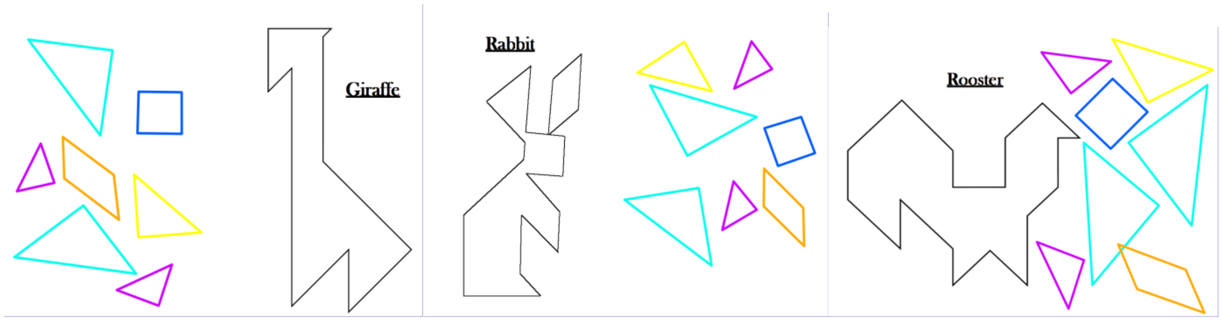

Figure 1: ***Pages 2-4 of Ms. Guider's geometry IWB center***

Each pair in the center also has one set of tangram manipulatives. The pieces are exactly the same as those shown on the board: two large triangles, one medium triangle, two small triangles, one square, and one parallelogram. By giving each pair of children a set of tangrams and asking them to solve the puzzle everyone can potentially be focused on the task rather than one pair watching another pair work. Once students solve the puzzle, Ms. Guider includes an additional voice recording that asks them to discuss which part of the puzzle was the hardest and why, and then asks them to switch roles (one pair works at the IWB while the other works with the concrete manipulatives). Ms. Guider implements this center over the course of a week. Groups of four students rotate through different centers throughout the week, visiting one center per day for approximately forty minutes. By the end of the week, all thirty of Ms. Guider's second grade students would visit the IWB center at least one time. Ms. Guider intentionally includes voice recordings within the IWB center to ensure that she does not have to be present to guide students through the task. This frees her to work with other groups of students on extending or reteaching additional mathematics content during center time.

Using the IWB in Algebra

This kindergarten center focuses on building algebraic thinking by examining, replicating, and extending repeating patterns. Mr. Bain, a kindergarten teacher, identified a need for this center when he saw students struggling to identify patterns during calendar time in the beginning of the day. Mr. Bain recognized that it is important to allow students to explore repeating patterns so they are better able to identify patterns in the world around them, a concept that is central to laying the foundation for later algebraic thinking (Van De Walle, et al., 2013; NCTM, 2000). To address these concepts, Mr. Bain designs patterns on the IWB in curved shapes rather than in straight lines (as patterns are often displayed) to encourage students to generalize their understanding of repeating patterns into different representations (see Figure 2). He had originally designed this task for whole group instruction with him directing the use of the IWB (as he had seen in other classrooms) but decided to restructure the task to increase collaboration and discourse amongst his students.

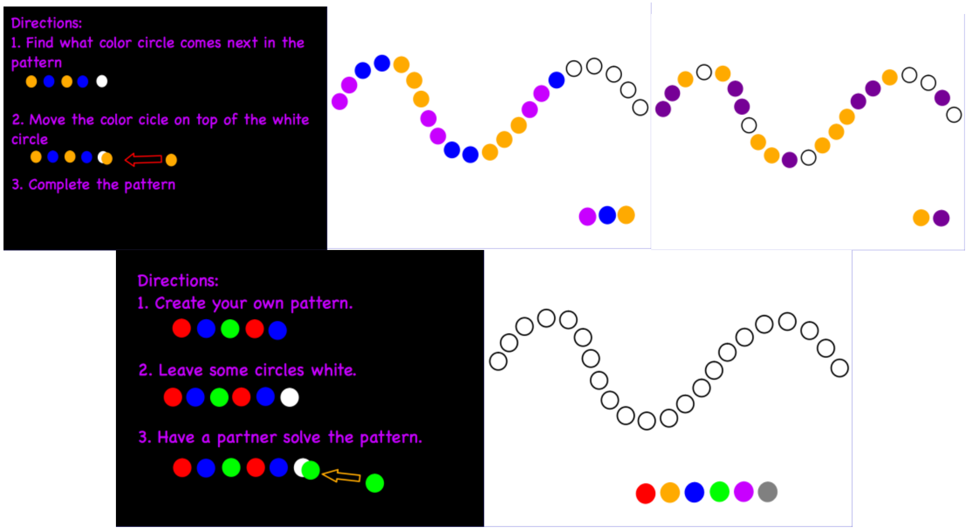

Figure 2: ***Pages 1-5 of Mr. Bain's algebra IWB center***

Mr. Bain encourages students to work with both the IWB and concrete manipulatives (pattern blocks) in the center to create and complete repeating patterns. In the beginning of the task, Mr. Bain asks a group of four students to extend a repeating pattern that had already been created. Then students are asked to fill in missing parts of a pre-established repeating pattern. Lastly, students split into two groups of two. Group One works to create a repeating pattern on the IWB, while Group Two works to create repeating patterns with pattern blocks on the rug in front of the IWB. After Group One finishes creating a repeating pattern on the IWB, Group Two has the opportunity to extend the pattern. As they work, Mr. Bain asks students to describe their patterns and to identify the parts of each pattern. He also asks groups to compare their patterns and describe how they were similar or different to each other. Following these questions, the two groups switch activities and continue their work with repeating patterns. By breaking students into two groups, each student can remain engaged in creating and completing repeating patterns during the entire center time. Students are placed in groups to encourage discussion about repeating patterns, which gives them the opportunity to better understand the mathematical concepts present in the center. Mr. Bain also implements this IWB center over the course of a week in which groups of four students rotate through a total of five centers in the classroom. By the end of the week, all students in Mr. Bain's kindergarten class visit the IWB center at least once. Because there is no voice recording embedded in the center, Mr. Bain stays close by to scaffold students as they work through the task by reiterating directions and prompting students to communicate with each other as they develop, extend, and describe patterns. He decides not to work with a separate group during center time to allow for flexibility in assessing students through observation or by asking open ended questions to understand students' thinking about the mathematical content as they progress through center tasks.

Discussion

These examples illustrate how IWBs can be integrated in mathematics lessons for young children through the use of classroom centers. Both centers meet the Common Core State Standards for Mathematics in terms of content and practice. The first example relates to the second grade geometry standard calling for students to "Reason with shapes and their attributes" (National Governors Association Center for Best Practices & Council of Chief State School Officers, 2010, p. 20). While there are no content standards explicitly relating to patterning in the kindergarten Common Core standards, the content in the second example does connect to the counting and cardinality standards (NGACBP, 2010) and the kindergarten algebra standards as described through NCTM (2000). In addition, and perhaps more importantly, the structure of the centers allow for students to meet many of the Common Core standards for mathematical practice, including the ability to "make sense of problems and persevere in solving them"; "construct viable arguments and critique the reasoning of others"; "model with mathematics"; "use appropriate tools strategically"; "look for and make use of structure"; and "look for and express regularity in repeated reasoning" (NGACBP, 2010, p. 6-8). Each example uses IWBs without compromising necessary characteristics that make a mathematical task effective:

1. *Building Communities*. Both IWB tasks encourage students to work in collaboration with each other either as a group of four or as a pair to examine the mathematical task. Students are never asked to sit and watch the teacher or another student demonstrate while working in the center. While at times, teacher demonstration is important for building understanding, in these scenarios, it is critical for students to be actively engaged as much as possible throughout the task.
2. *Increasing Discourse*. Structuring tasks so that students work in pairs and are given specific roles provided opportunities for increased discourse that may not occur when the IWB is used in a more teacher directed context. In the geometry example, Ms. Guider asks questions and gives directions through an electronic recording, which provides students with some structure for engaging in discussion within the task. In the algebra example, students are asked to extend each other's patterns as they work in pairs. By giving students a specific role to fulfill, they are more prepared to engage in discussions about their patterns. In addition, by deciding not to work with a separate small group, Mr. Bain has the flexibility to be present in the center to ask open ended questions about the pattern that can encourage discussion.
3. *Using Materials*. In both examples, it is important to include concrete manipulatives for students to work with as they engage in the IWB task. The tasks are structured on the IWB in a way that is mathematically meaningful, however, manipulating the material on the IWB can be difficult for some young children who need to pick the material up and examine it from multiple angles to solve the problem. By having additional materials present, students can be actively engaged throughout in a way that was mathematically meaningful to them.
4. *Making Connections*. Both teachers attempt to structure the center tasks to ensure that students can make connections between the content and their own experiences or between the content and what their peers are doing within the task. In the geometry example, students are asked to examine how shapes fit together to create pictures of commonly identifiable animals (rabbit, giraffe, rooster) that children may have read about at home or in school settings if they have not encountered the animals in real life. In the algebra example, students are asked to describe their patterns and compare their patterns to those of their peers.
5. *Focusing on Processes*. Both examples are structured around problem solving and reasoning. Students are not asked to use one specific strategy for solving the tangram puzzles in the geometry center; they have to reason about the shapes they are given to determine how they fit together to make the animal. While students are asked to extend patterns in the algebra center, they are also asked to represent and extend their own patterns.

Conclusion

The examples presented in this article demonstrate that the creation and implementation of center based mathematical tasks does not need to be too technologically difficult, rather, in the early childhood classroom, the more streamlined the task, the easier it will be for students to complete without assistance. In these instances, the use of an IWB has the potential to extend learning for students within the context of a center by providing an engaging tool for which students could work together to manipulate and complete a specific mathematical task without continuous teacher direction.

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Mathematical Inquiry: An Instructional Model and Web-Based Lesson-Planning Tool for Creating, Refining, and Sharing Inquiry-Based Lessons

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Abstract

One means of reforming mathematics instruction is through mathematical inquiry. This paper presents the 4Ex2 (read 4E by 2) Instructional Model for inquiry-based instruction and illustrates the model's four phases with a dilation investigation in which students explore geometry, measurement, and proportional reasoning. A dynamic, web-based lesson-planning tool for creating, refining, and disseminating inquiry-based lessons using this framework is described.

Educators have called for reform in mathematics teaching and learning for quite some time, yet in many classrooms instruction has only marginally moved away from traditional practices of teacher-worked examples followed by students repeating procedures void of context. Perhaps one reason for this slow transition is that the reform movement, although grounded in constructivist theory, does not commit to a specific teaching method. Mathematical inquiry is, nonetheless, one instructional method congruent with the goals of the reform movement (Horton, Sloop, & Marshall, 2014).

To help teachers construct, modify, and share inquiry-based lessons, math and science educators at Clemson University created a web-based, lesson-planning tool based on the 4Ex2 (read "four E by 2") Instructional Model (Marshall, Horton, & Smart, 2009) for content-embedded, inquiry-based instruction. The purpose of this paper is to first describe the 4Ex2 Instructional Model illustrated with an inquiry-based mathematics lesson and then to show how the Inquiry in Motion web-based lesson planning tool helps educators create, refine, and disseminate inquiry-based lessons.

The 4Ex2 Instructional Model

Contrary to traditional educational practices where teachers take the desired content, break it into palatable portions, organize the material in a linear manner, and give a number of examples, constructivists view students as meaning makers who learn best when they are actively organizing content while making and testing conjectures. This sense making occurs when students attempt to bring newly acquired information into equilibrium with their current understanding of the world (Piaget, 1975/1977). The 4Ex2 Instructional Model is based on work in science education (Bybee, 1997; Eisenkraft, 2003; Marshall, Horton, & Smart, 2009) and founded on the tenet that students must first experience situations that cause some type of mental disequilibrium (Piaget, 1961) before material can be consolidated and organized. That is, students should *explore* the content before it is *explained*. However, students may become frustrated if heedlessly thrown into this state of disequilibrium without attending to their previous experiences and current understandings. Therefore, it is helpful to first *engage* students in meaningful mathematical tasks while assessing their prior knowledge. After students' interests are engaged and they explore problems to form an explanation of the mathematics, students are challenged to generalize or apply their results and *extend* their findings to new contexts. The 4Es of the 4Ex2 Model, illustrated in Figure 1, represent the four phases of mathematical inquiry: Engage, Explore, Explain, and Extend.

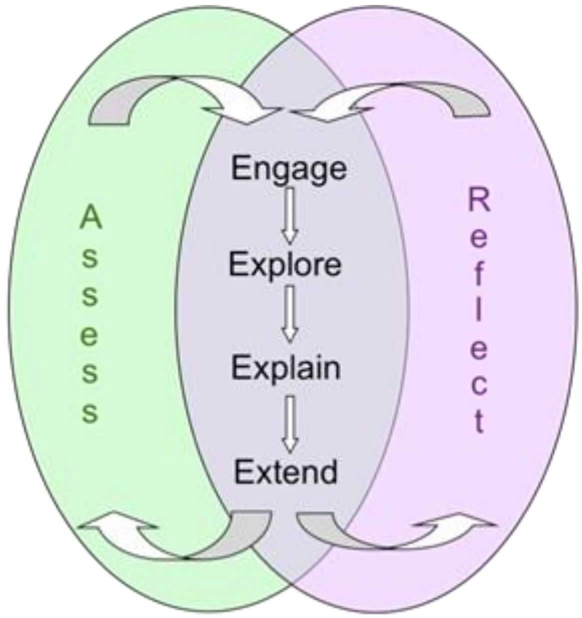

***Figure 1*: 4Ex2 Model for inquiry-based instruction**

While it is important that students explore before an explanation is given, the 4Ex2 is a dynamic model where formative assessment and teacher reflection determine the instructional path. Perhaps a teacher notices from students' explanations that further exploration is needed before entering the Extend phase. This lesson might follow a sequence of Engage, Explore, Explain, Explore, Explain, and then Extend. It is the abovementioned formative assessment and reflection during each phase of inquiry that is represented by the x2 in the 4Ex2 Model.

In the section that follows, each phase of the 4Ex2 Model is described and illustrated with an inquiry-based lesson in which students investigate proportional reasoning while exploring dilations. In most dilation activities seen at the middle-school level, students multiply points in a coordinate plane by a scale factor to dilate an image about the origin; however, in this investigation, students will use rulers, compasses, and protractors to dilate a figure about a fixed point.

Engage

In the 4Ex2 Model, students begin their inquiry experience in the Engage phase. During this time, the teacher may pose a mathematical task or provide experiences such that students pose problems themselves. It is during the Engage phase that a context is established for content to be addressed. In addition to piquing students' motivation and interest, this is an important time when the teacher uses formative assessment to gauge students' prior knowledge and to identify alternative conceptions. During this phase, students may develop questions to explore further, make conjectures or predictions, and devise a plan to explore the mathematics. The Engage phase is a time when the teacher is encouraged to reflect upon students' interests, prior knowledge, and current understandings and then determine how she or he might tailor the next phase to the students' needs, perhaps in part adapting instruction as the evidence suggests.

In the Engage that follows, the teacher may introduce the mission of the American Cancer Society (ACS) or have their local ACS community manager speak to the class about how Relay for Life (RFL) raises money for cancer research. The teacher or community manager will then request the assistance of the class in making Relay for Life banners for their school's event. Student will be given a copy of the RFL logo, illustrated in Figure 2, and work in groups to construct a rough draft of their banner on butcher paper using protractors, rulers, and compasses. Students will be challenged to use the measurement tools provided to identify patterns between the printed logo and their banner. The teacher will also challenge students to organize their findings using a number of representations to present to the class.

***Figure 2*: Relay for Life logo**

Groups will present their drafts and explain their strategies for dilating the logo. During this discussion, the teacher will probe students to explain how and why they used the geometric tools provided to preserve shape and location of the pieces of the logo while assessing students' prior knowledge of proportions, geometry, measurement, and representing data.

The teacher may try to reproduce students' strategies on the board as they explain, exaggerating potential mistakes from unclear procedures. For example, if students are unclear in their presentation of how they determined where to place the star, the teacher might purposefully place the star in an obviously incorrect position to emphasize the need for a systematic dilation procedure.

Students will also present the tables of data and graphs they constructed of various measurements between the logo and dilated banner. If students have not already identified the scale factor, the teacher will question students in order to connect this ratio with the unit change (slope) of the data they collected. Teachers may also choose to address the distributive property when questioning students about whether the perimeter of certain shapes increases with the same proportion.

Explore

In the Explore stage, teachers provide mathematical experiences for students and challenge them to use these experiences to make sense of the mathematical task. This is a time when concrete models might be used to explore relationships, data could be collected in search of patterns, or conjectures could be tested. During the Explore, students select appropriate problem-solving strategies or construct new strategies to investigate the mathematics. As students explore together, meaningful mathematical discourse is encouraged as students analyze each other's strategies and generate new questions that can be addressed in an Extend phase. Teachers should use formative assessment to evaluate students' strategies for designing methods to test conjectures, collecting data, or justifying their findings. Teachers should also reflect on whether students are ready to explain their findings, whether further exploration is needed, and the amount of scaffolding that is appropriate.

In the Explore phase of the lesson presented here, students will use dynamic geometry software to construct the RFL logo over an image of the logo inserted into the sketch. From here, students can hide the original image and use the measurement utility of the software, illustrated in Figure 3, to determine angles between figures, radii of circles, and lengths of segments used to determine the proportions and placement of various figures in their dilation. Students will then use these measurements, rulers, compasses, and protractors to create their banner.

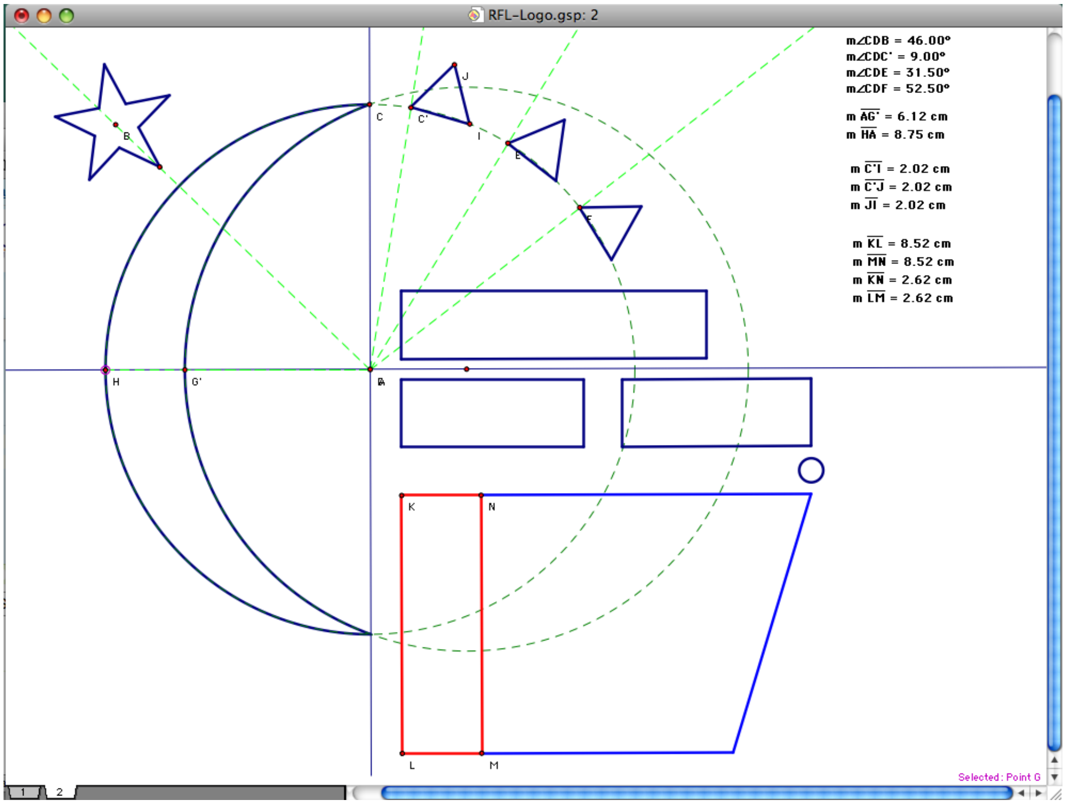

***Figure 3*: Exploring RFL logo using dynamic geometry software**

Explain

With shared mathematical experiences, students are ready to synthesize their understandings by communicating their results. During the Explain phase, students are encouraged to organize their data and interpret their findings. Students may use a variety of representations to communicate mathematical ideas. Group presentations are one formative assessment that encourages students to evaluate and analyze alternative explanations, solution strategies, and mathematical reasoning. With explanations coming from peers, as opposed to the teacher, rich student-to-student discourse promotes autonomy in problem solving. During the Explain phase, teachers might reflect on whether students mastered the desired content, were able to make sense of the data they collected, and whether students provided convincing evidence for their claims. Depending on the objectives and the standards the teacher has selected for the lesson, attention should be paid not only to what the students have found, but why they have found what they did. Thus, mathematical reasoning, which should stem from the students as much as possible, is essential to real sense-making.

Although students justified their strategies to group members throughout the Explore, in the Explain phase of this lesson, they will more formally present their work to the class explaining both how they constructed their "blue print" and which measurements they used to ensure the proportions and positions of figures are preserved. The class will critique each group's design to ensure they had enough information to create a proportional reproduction similar to how the teacher exaggerated vague dilation procedures in the Engage.

Extend

The Extend is a time for students to investigate further questions that might arise during the Explore or Explain. It is here that students generalize patterns discovered previously. Students might apply the mathematics to a different context or elaborate on their conclusions. Students could be challenged to prove their conjecture holds in different conditions or explore the assumptions on which their conclusion is founded. During the Extend phase, significant connections are made among mathematical concepts and to other contexts. In the Extend, formative assessment allows the teacher to measure students' ability to generalize, transfer, apply, and elaborate. If the Extend is the final phase of the inquiry, it would be appropriate for the teacher to reflect on the lesson as a whole. She or he might consider how the lesson could be improved and what changes should be made.

Although a variety of mathematical tasks could extend the ideas of measurement, geometry, and proportional reasoning explored in this lesson, one possible Extend would be to examine how dilating this figure affects its area. The teachers may ask students to calculate the exact amount of fabric needed to construct a RFL flag, the minimum practical amount required, or have students create a tessellation of figures to minimize the amount to fabric needed. Then, students will be encouraged to consider the relationship between the area of the dilated image, the area of the original figure, and the scale factor of their dilation. The teacher should scaffold students toward discovering that the area of the dilated figure is proportional to the area of the original figure with a ratio equal to square of the scale factor.

Creating Inquiry-Based Lessons

Due to the dynamic nature of the 4Ex2 model, an equally accommodating electronic planning tool is required. The Inquiry in Motion (IIM) lesson planning tool (located at www.clemson.edu/iim) provides a template for designing inquiry-based lessons with the structure needed to keep content-focused guided inquiry in mind while allowing the flexibility required by this dynamic model.

When creating inquiry-based lessons, it is important to keep content central. The IIM planning tool allows the teacher to quickly and easily select from national content standards as well as specific South Carolina, North Carolina, and Common Core Standards for both as seen in Figure 4. For educators in other states, customized state standards can be, and currently are being, added using the Customized Standards tab seen in Figure 4. Given the 4Ex2 model's emphasis on formative assessment to gauge prior knowledge, teachers are prompted to consider pre-instructional requirements, such as the prerequisite knowledge that students need before they will be able to engage successfully in the lesson.

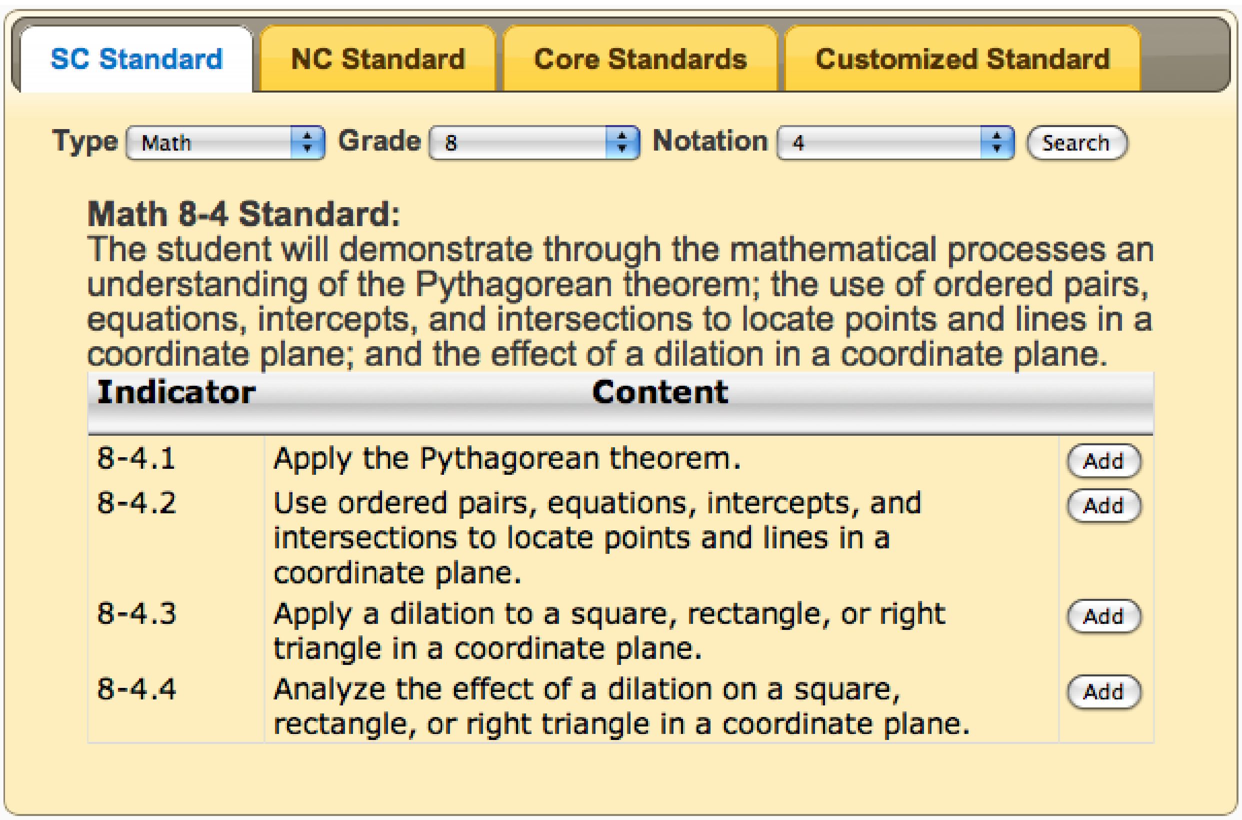

***Figure 4*: Selecting standards for an inquiry-based lesson**

When creating an instructional plan, the teacher can add as many Engage, Explore, Explain, and Extend phases as appropriate. The teacher can also readily reorder these phases as desired, attending to the dynamic nature of the model. Within each phase, the user is prompted to identify specific sub-components she or he wishes to address. For example, when adding an Extend phase, the teacher chooses whether the goals for this piece are to apply, elaborate, transfer, or generalize as seen in Figure 5.

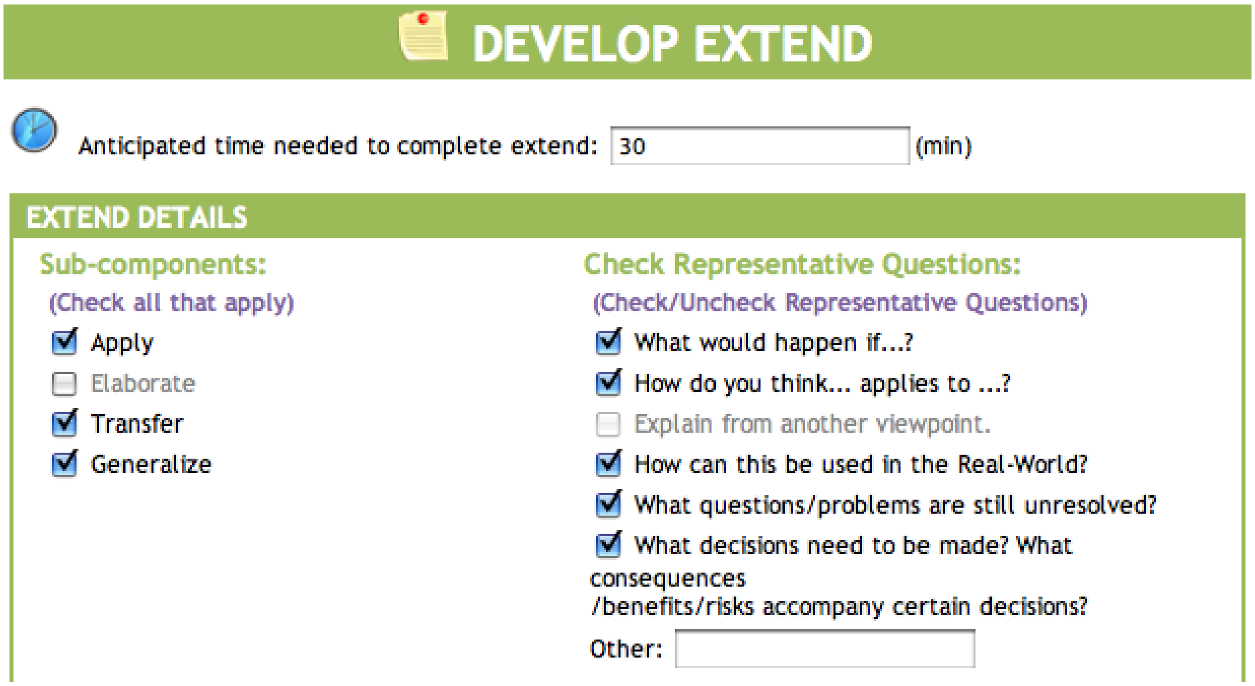

***Figure 5*: Selecting sub-components and identifying questions for reflection**

With the model's emphasis on reflection, a list of representative questions is suggested based on the sub-components chosen. Other representative questions can then be selected from a list, or new questions can be added. Similarly, a list of authentic, formative assessments appropriate to a particular phase is offered that teachers can select; they also have the option of adding other forms of assessment, as seen in an example in Figure 6. While the planning tool offers pre-test and think, pair, share—to name two of several options—as compatible assessments for an Engage, a teacher might choose to add a journal entry as a means of formative evaluation.

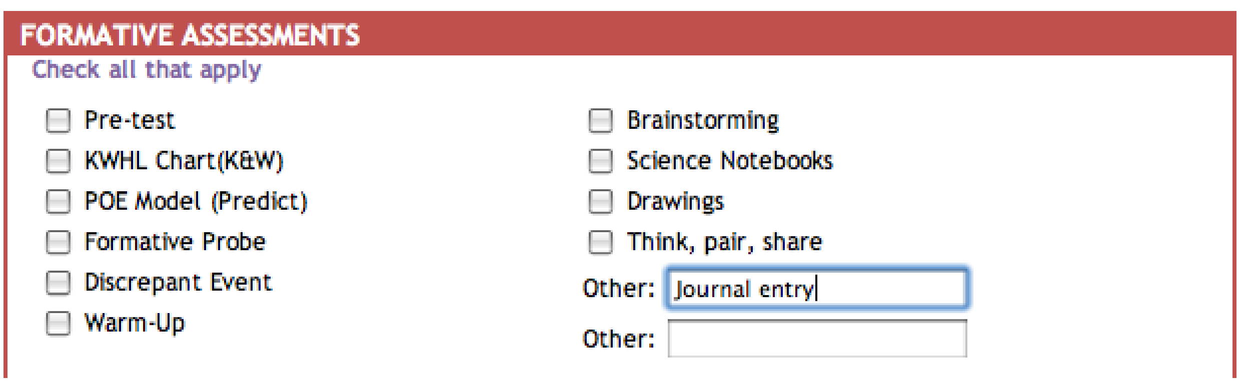

***Figure 6*: Selecting formative assessment for an Engage**

When attempting to design lessons using a new model, a teacher may find it difficult to create content-embedded, inquiry-based lessons in isolation from others with similar goals. Sadly, it would not be uncommon for a teacher to find her- or himself to be the only educator in a particular school advocating for inquiry-based reform practices. Therein, the IIM planning tool allows teachers to connect and collaborate with other math and science educators with similar goals. When creating a lesson, teachers can add additional authors. With this feature, teachers can work with other educators from a distance to create lessons of a quality not possible in isolation.

Through focus groups with teachers using the IIM planning tool to create lessons using the 4Ex2 Model, teachers identified a number of ancillary features that would aid them as they use this tool. Consequently, a rich text editor with Microsoft Word pasting compatibility, an embedded hyperlink utility, and a spell-checking feature were added. Teachers can also upload supporting documents such as quizzes, rubrics, and interactive whiteboard presentations. For example, for the Proportion for Life lesson presented here, the RFL logo and dynamic geometry software files are provided.

Refining and Modifying Lessons

In addition to the lessons posted for public viewing and use by anyone that visits the site, teachers who register for a free account also have a place to improve or adapt existing lessons. Teachers can select suitable lessons from those posted publicly and copy these lessons to My Workspace. Once a teacher moves a lesson to My Workspace, she or he can edit the lesson to best serve the population at hand. For example, a special education teacher might modify a lesson to be more congruent with her or his students' specific exceptionalities. A teacher working with 50-minute periods might adapt a lesson originally designed for 90-minute blocks (though the lessons built on the website are not confined to a single period). Since the meaningful mathematical tasks posed in inquiry-based lessons often address a number of standards in a single lesson, a teacher may find that a lesson originally intended for one standard could be adapted to emphasize another standard.

Disseminating Lessons

Given the number of lessons available online, teachers may find sorting through these lessons in search of an inquiry-based lesson focused on specific content to be a daunting task. The IIM planning tool, however, provides a repository for high quality, content-embedded, inquiry-based lessons. Teachers can quickly and easily search the ever-growing collection of lessons by title words, subject area, grade level, or national standard.

A major weakness of most lesson planning sites available on the Internet is the lack of quality control for free, teacher-generated lessons. In addition to the structure the planning template provides, the IIM planning tool ensures the quality of its lessons by allowing for evaluations by the math and science educators at Clemson University who serve as administrators of the system. These administers can assign silver and gold distinctions to lessons by evaluating the lesson on 10 essential and 10 important criteria listed in Table 1. Lessons that meet all 20 criteria earn the gold certification while those that meet all 10 of the essential criteria and at least 8 of the 10 important criteria earn the silver certification. System administrators can also provide feedback when evaluating lessons, which is automatically emailed to the authors. Through communication with those familiar with the 4Ex2 model, teachers can continue to improve their lessons.

Essential (All criteria must be met to receive gold or silver certification.)	
1	National standards are clearly specified and aligned with state standards.
2	Sufficient background information is provided.
3	The instructional plan effectively addresses the national and state standards specified for this lesson.
4	Sufficient detail is given for each phase on the instructional plan to allow another teacher to duplicate this lesson
5	Explore phases of the instructional plan precede explanations of content.
6	Lessons are largely student-focused with students taking an active role in learning.
7	Real world or meaningful context is embedded in the lesson to promote conceptual understanding.
8	Safety issues are addressed as necessary.
9	The lesson addresses a fundamental concept or big idea in either math or science.
10	Lesson is cohesive: standards, lesson, and assessments are well-aligned.
Important (Gold requires all 10 criteria; silver must meet at least 8.)	
1	A concise lesson overview is provided.
2	State standards are identified by number and description.
3	Prerequisite knowledge is addressed as necessary.
4	Necessary materials are listed.
5	All necessary teacher support documents are present and labeled appropriately.
6	The Instructional Plan includes engage, explore, explain, and extend.
7	At least one sub-component (e.g., prior knowledge) is specified for each phase of the instructional plan.
8	Formative assessments that will guide instruction are provided and described for each phase of the instructional plan
9	Questions for teacher reflection are present for each phase of the instructional plan.
10	The lesson makes connections to other concepts in math or science.

Table 1: Essential and Important Criteria for Evaluating Lessons

In addition to the IIM planning tool's features that allow teachers to locate reviewed lessons, this site offers other supporting features that aid teachers attempting to share their work. The planning tool allows teachers to stream media from the IIM site and download supporting documents. For teachers who require a hard copy of a plan, lessons can be converted to a printer friendly portable document format (PDF). Samples of students' work and video clips of teachers implementing these lessons can also be added to the lessons.

Conclusion

Although mathematics educators have called for reform in instructional practices for years, progress away from traditional practices towards student-centered learning founded on constructivists' learning theory has been slow. This slow evolution of instruction can be attributed in part to vague understandings of reform and an uncertainty regarding how one might implement such practice. The 4Ex2 model for inquiry-based instruction provides a framework for teachers to create and implement rich content-focused lessons. The IIM web-based lesson-planning tool assists teachers by providing a template for creating lessons using this model, allowing teachers to continually refine and improve existing lessons, and offering a repository for the dissemination of high quality inquiry-based lessons.

This model and planning tool provides a venue to unite teachers with a focus on inquiry-based instruction as they create, refine, and share lessons. With these tools, the number of accessible high quality, inquiry-based lessons will, we hope, increase.

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Fact Fluency:
Reasoning before Flash Cards and Speed

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Abstract

The CCSSM and the NCTM PSSM reference the need for children to be fluent and efficient with math facts. However, fluency and speed are separate skills often treated as one. This article provides support for teachers in moving toward the use of reasoning strategies for instruction and fluency prior to focusing on speed. Two specific activities are presented with discussion on their use in the classroom, differentiating instruction, and helping children gain mastery.

Can learning the basic facts be fun and engaging? Yes, it can! Automaticity is important for students to be able to spend more of their mental problem solving skills on new information in lieu of spending time to determine simple math computations (Woodward & Montague, 2002). Automaticity, or fluency in mathematics, is referenced within both the National Council of Teachers of Mathematics (2000) *Principles and Standards* as well as the Common Core State Standards for Mathematics (CCSSM) (CCSSO, 2010). Fluency in these cases can be defined as a student's ability to make computations "efficiently and accurately" (NCTM, 2000). Additionally, the CCSSM (CCSSO, 2010), beginning in grade 2, specifies that children will be able to "use mental strategies" and "know from memory" sums of two one-digit numbers (p. 19). Learning basic facts through the use of flash cards, timed tests and methods that show "public comparisons of mastery" (Van de Walle Karp & Bay-Williams, 2013, p. 188) can be anxiety inducing and reduce mathematics self-efficacy. So, how, then, can we help children develop fluency in mathematics so they can devote brain power to the real math problem instead of trying to determine the answer to the basic fact?

According to Baroody (2006), children move through three distinct phases when learning math facts. During the first phase of fact mastery, children must establish a mastery of counting (object counting as well as verbal counting). Phase two is identified by the development of reasoning strategies to determine the answer to a combination of numbers or an answer to an unknown. Baroody, Bajwa, and Eiland (2009) have indicated that phase two has been neglected by an immediate jump to the memorization of facts in isolation or by fact families. However, it is after children have mastered reasoning strategies that phase three, mastery and subsequently, speed, can take place. Baroody et al. (2009) concur that "mastery with fluency grows out of the development of meaningful and well-interconnected knowledge about numbers – number sense." (p. 70) In fact, those with a passive storage view on teaching fact fluency focus on the rote memorization (Baroody et al., 2009) of the facts which Woodward and Montague (2002) point out allows children to "learn the procedures without any conceptual understanding" (p. 95). On the contrary, those with an active construction view on teaching the basic facts, also referenced as the Number Sense View, believe children develop fluency as a result of building phase 1 and phase 2 skills.

There are multiple methods for assisting students in developing strategies for mental mathematics. In keeping with Baroody et al.'s (2009) focus on the Number Sense View educators need, first, to help children create their own thinking strategies by:

1) Familiarizing them with multiple methods for representing the facts,
2) Finding patterns among the facts, and,
3) Encouraging self-invented strategies for solving the facts.

Student self-invented strategy development connects with the CCSSM Mathematical Practice (MP) 7: Look for and make use of structure. The children, by recognizing patterns and mentally creating relationships for number combinations, utilize structure of number to become efficient in solving the basic facts. It could also be argued that MP 8: Look for and express regularity in repeated reasoning, is also apparent when children begin to reason through mental shortcuts in solving problems. Students, who can mentally strategize to develop their own methods for solving problems, quickly substitute mental shortcuts once reasoning strategies are established.

The remainder of this article includes two useful reasoning strategies and activities for educators to develop students' abilities in understanding and developing fact mastery in addition. The first strategy creates a mental picture of number and addition combinations, which activates student prior knowledge. Activating prior knowledge helps not only in reasoning but also in developing student efficiency with the facts – the ultimate goal toward obtaining speed. Developing speed indicates a level of mastery in which students show their efficiency by being able to quickly answer a fact by, as Baroody (2006) notes, "Just knowing it." The strategies offered within this article provide fun, non-anxiety inducing methods for helping students gain speed in lieu of the timed test of facts.

In order to develop mental mathematics fluency, teachers must: (1) show numerous strategies, even those strategies teachers may not be comfortable using themselves; (2) encourage self-invented strategies; (3) listen to students explaining the strategies they've used or developed; and, (4) allow children to share strategies with one another. Van de Walle et al. (2013) also indicate that it is imperative to allow student invented strategies since invented strategies require student understanding; thus, showing the teacher he or she is prepared for fact mastery.

Flash Manipulatives - Not Flash Cards

Direct-modeling through the use of base-ten blocks, ten-frames and pictorial representations assist children with developing a visual of what the combination or removal numbers looks like. For children who have used ten-frames to learn counting, the use of the frames creates a consistency with methods they have learned before. Thus, a confidence and comfort level with learning a new mathematical concept can be established. These children would have already gained a visual of number through their counting strategies with the ten-frames and can more efficiently use counting-up or counting-on strategies. The following activity is a ten-frame flash followed by using ten-frames to make 10. The ability to make 10 not only develops fact mastery for those facts that add to 10 but also develops children's further understanding of facts that are more than 10.

Figure 1 shows two ten-frames with differing numbers of place counters. This initial activity helps students to anchor to values of five and 10. Thus, creating a mental picture for students and providing opportunity for reasoning of numbers that create five and make 10.

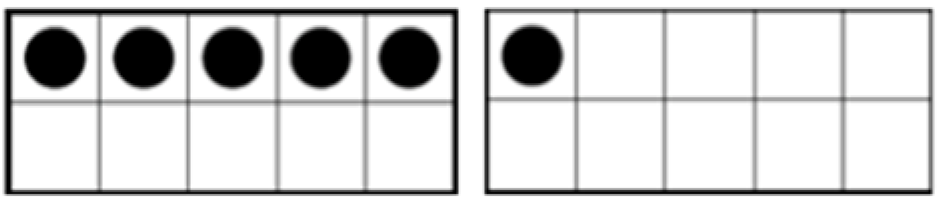

Figure 1: ***Ten-Frame Flash – Activating Prior Knowledge***

To activate prior knowledge, the teacher first practices placing one ten-frame in front of students, flashing the frame for about five seconds and then re-covering the frame. The teacher should ask students to report how many counters they saw and ask how they know. The teacher can also request students recreate what they saw as a differentiation strategy, if needed. It is important to note, prior to moving on beyond five, teachers should first begin with placing counters up to five, repeating these flashes for students struggling with fast identification. Quick identification shows the number is well established within the child's mind.

By first developing number relationships, children acquire the initial strategies and understandings that are needed for fact mastery. Viewing two ten-frames, one with 5 counters and one with 1 counter, students create a mental image of 5 and 1. The teacher first shows the combination of the two frames as is shown in Figure 2, and follows by showing students the final sum with the resulting picture as is shown in Figure 3. These mental images will help students memorize their basic facts to ten and allows them to create mental strategies for recalling facts more than ten such as that presented in Figure 4. Students can easily quiz themselves and each other on basic facts. First, students begin with flashing ten-frames and practicing the transfer of counters to numbers mentally; then, by showing two ten-frames one above the other with the value for each represented and practicing the combination of the two mentally.

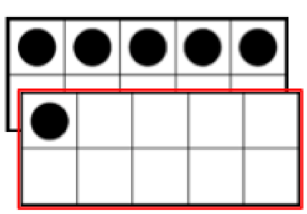

Figure 2: ***Mental Image Combining Strategy***

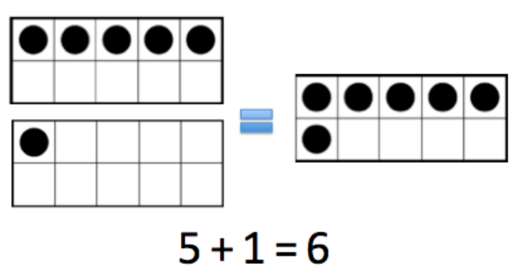

Figure 3: ***Final Mental Image***

Reasoning and Fluency Beyond 10

The ten-frame flash activity moves children from the Phase 1, counting level, into the Phase 2, reasoning level. A natural follow up to the ten-frame flash strategy is the Break Apart to Make 10 (BAMT) (Sarama & Clements, 2009) model. The BAMT is a strategy utilized by many high-achieving countries, but not highly emphasized in U.S. classrooms and textbooks (Van de Walle et al., 2013). The children in the high-achieving countries that emphasize the BAMT strategy learn their facts at a much quicker pace with more accuracy than U.S. children (Van de Walle et al., 2013). The BAMT strategy covers approximately one-third of the addition facts and as such, Van de Walle et al. (2013) recommend that this reasoning strategy become more significantly used and focused upon in U.S classrooms.

The BAMT model addresses facts with sums greater than 10. Children apply the foundations learned previously for facts up to 10 to build upon their knowledge for facts with sums more than 10. So, in the problem 9 + 6, a child would take one from the 6 to make 10. That would make 10 + 5. The child would quickly come to the conclusion that 9 + 6 = 15. Figure 4 shows how a teacher might introduce this concept using ten-frames. Further, this strategy represents a clear example of children showing fluency in the use of MP 7 and MP 8.

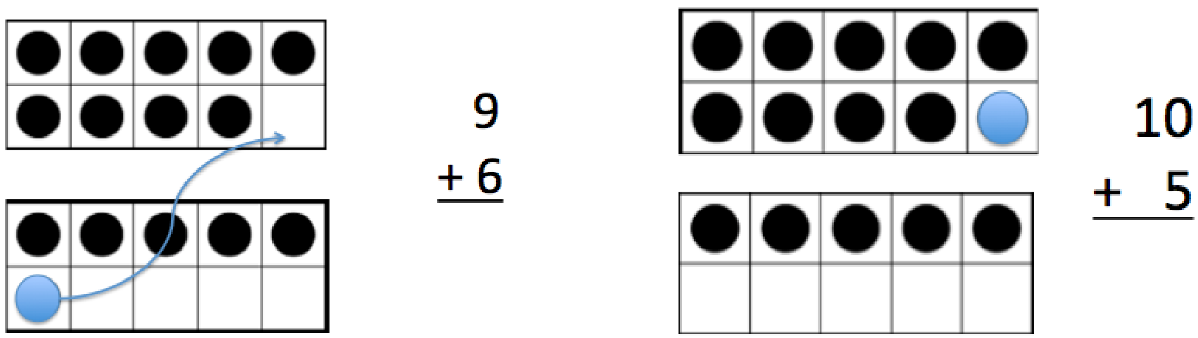

Figure 4: ***BAMT Reasoning Strategy for Sums More than 10***

Fact Frames for Practice and Mental Math Fluency

After teaching both of the ten-frame based activities, teachers can make (or purchase) cards with differing combinations of ten-frames with the numerical fact representation next to it. The cards would look like the examples given in Figure 4. Children are given two frames and the math fact to be learned at the same time. Thus, children have a self-correcting card (no answer is needed), and differentiation is automatic allowing for children at different levels of mastery. This type of flash card allows children to count or work faster toward mental mathematics mastery depending upon where the child may be in the transition from Phase 2 to Phase 3.

Additionally, children that are struggling with memorization of the facts will not benefit from continued drill. A return to the foundational skills of mental mathematics will need to take place prior to continued drill. Differentiating instruction for struggling facts learners is imperative for reducing anxiety and supporting confidence in mathematics ability. Not all children will learn from a strategy like the one shared above. For this reason, educators must present and encourage multiple types of reasoning strategies.

Final Thoughts on Drill & Practice

As was mentioned before, drill can be appropriate at times. However, a teacher must first consider which children have substantial reasoning of number relationships and are ready for speed. Children can quickly become overwhelmed by too many facts at one time; thus, timed tests aren't necessarily the way to go. However, Van de Walle et al. (2013) suggest games and a focus on self-improvement will work to help children develop automaticity. Additionally, such activities as the Ten-Frame Flash are a type of drill, which can be done in short spurts and should not induce panic among students. Although the frame flashes are simple, they result in automaticity without the use of a timed test. While, in my opinion, flash cards are unnecessary, teachers and families who like to utilize flash cards should use only a few at a time. One set of facts should be well learned before moving on to the next. While using the cards, children can also track their own progress using a facts matrix and coloring each fact he/she has mastered. Thus, the child, teacher, and family can see the progress the child has made and celebrate the child's growing knowledge.

Drill and practice of the basic facts can be made fun! Using games and technology to reinforce the facts allows for children to associate mathematics with fun rather than anxiety. While drill is considered a non-problem solving activity, it can be made fun by allowing children to work together at a center or during a specified game time to address fact speed and reasoning. As long as children aren't made to feel that their inadequacies are put out for all to see during the game, such as the "Around the World" math facts game, games are engaging and non-threatening. Additionally, games that require students to self-monitor and provide a control of error, such as math-based dominoes games increase engagement and enjoyment. Moreover, engagement and enjoyment will lead to achievement. A last thought to leave you with…have fun WITH your students! In working with in-service and pre-service teachers, basic facts practice with games and cards is fun for adults too! You may be surprised at how much practice you, as an adult, might need on your basic facts!

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Implementing Universal Design for Learning Principles Using Mobile Technology

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Abstract

This study explores the use of the iPod Touch in a fifth-grade math classroom to increase achievement and student engagement. The iPod Touch provides a variety of resources related to Universal Design for Learning (UDL). This study applied the elements of UDL in math lessons on equivalent fractions. To facilitate its implementation, participating teachers developed and implemented technology-integrated lessons. Over 90% of students in the experimental group expressed high positive motivation towards iPod use.

The architectural movement in the 1970s, termed “universal design” by Ron Mace, promoted the idea of designing buildings from the outset to be accessible to everyone. The idea was twofold: (1) it was less expensive to design accessible buildings from the onset rather than retrofit the building later, and (2) everyone could benefit from accessibility features such as elevators and curb cuts, not just people with disabilities. According to Rose and Gravel (2010), from this movement arose the concept of Universal Design for Learning (UDL) in education, “a framework for teaching and learning that often capitalizes on the power and flexibility of modern technologies to address the needs of the broadest possible range of students” (p. 1). Again, the idea is that rather than develop a lesson plan and then go back and provide accommodations for various types of learners, develop a lesson from the beginning that incorporates features that will help all students learn. These strategies can include using graphic organizers and providing visuals in addition to text to explain a concept. Two characteristics of digital multimedia make it ideally supportive of UDL, its versatility and its flexibility.

UDL encompasses three principles common to many perspectives in the learning sciences: (a) providing multiple means of representation, (b) providing multiple means of action and expression, and (c) providing multiple means of engagement (Rose & Gravel, 2010). However, the critical focus of UDL is its emphasis on the variables that can be manipulated to produce high performance, or expertise (Edyburn, 2010). An example is providing different ways (e.g., physical, sensory, and cognitive) for retrieving information that are “accessible at a level of appropriate challenge” (Edyburn, 2010, p. 40). In choosing how such variables are operationalized, the instructor must closely attend to aligning the instruction with the goals of the learning experience (Sears-King, 2009). To these ends, the current study explores the use of the iPod Touch in a fifth-grade math classroom to enhance achievement and student engagement.

The iPod touch provides a variety of resources that can be used to differentiate instruction and are related to UDL principles (Keengwe, Pearson, & Smart, 2009). The following table provides examples of how the iPod touch can be used in math to enhance representation, expression, and engagement.

Multiple Means of Representation	Multiple Means of Expression	Multiple Means of Engagement
Teachers and students can use the iPod touches to locate and view videos related to specific course content. Teachers can use the audio and video functions of the iPod touch to record lectures or portions of lectures so that students can review course content as needed.	Students can create videos related to course content. Students can create podcasts related to course content. Students can record their reflections on their own work. Students can record their thought processes while engaging in the scientific method or solving mathematical problems (think aloud).	Students can practice specific skills using software downloaded on the iPod touch. Students can work in groups to create digital videos or podcasts. Students can review content delivered and recorded by the teacher if they miss class or need to review material.

Another purpose of the present study is to establish ways to measure the effectiveness of UDL for enhancing the academic performance of diverse students. According to Edyburn (2010), "UDL outcome measurement needs to focus on the benefits that result from access and sustained engagement: expertise and expert performance. Ultimately, we need to understand how to measure the contributions of UDL to sustained engagement and development of expertise" (pp. 39-40).

Research Project

This pilot study applied the elements of Universal Design for Learning (UDL) in math lessons with the goal of increasing student achievement and engagement. To facilitate the implementation of UDL, participating teachers at a small, rural, Title I elementary school in South Carolina developed and implemented technology-integrated lessons using iPod touches.

Research questions for this action research study included: (1) Will introducing UDL elements in math using mobile technology, i.e., the iPod touch, increase student achievement?; (2) What are the perceptions of the teachers regarding using the iPod touches?; and (3) What are the perceptions of the students regarding using the iPod touches?

Materials and Methods

We purchased the following materials: 10 iPod touches, 10 iPod touch covers, 10 iPod touch screen protectors, 20 headphones, 10 headphone audio splitters, and multiple port USB chargers. The iPod touches were cataloged through the school media center check-out system. Teachers checked out the devices from the media center if they wanted to use them. We also purchased a small file cabinet that locked to store the iPods in the classroom when they were not in use.

The project was implemented in two fifth-grade classrooms, one serving as the experimental group(used iPod Touch) and one serving as the control group (used Everyday Math game). Both classes used Everyday Math for the instructional sequence. Teachers at the school chose to teach five lessons in math. Each lesson was the same lesson from the Everyday Math series. Teacher “A” used the iPods to reinforce the lesson and allow practice. She chose and provided all students access to Every Day math applications (apps) such as “Top-it” and “Equivalent Fractions.” Teacher “B” used a card game also provided by Everyday Math for reinforcement and practice. A posttest on the indicators covered in the lessons was administered using Flanagan’s assessment instrument, a standards-based multiple choice instrument aligned with the South Carolina Curriculum Standards. Students in the experimental group were also surveyed by the teacher about their engagement in the lesson. The university faculty-in-residence at the school observed one lesson in both the control and experimental situations and verified that the instruction was similar and covered the same objectives. The major difference was the treatment, the elaboration of the instruction using Everyday Math apps on fractions on the iPod touch in the experimental group and using an Everyday Math card game in the control group. Both approaches can be considered applications of UDL principles. The iPod Everyday Math apps presented information to students using images and audio. The students were also allowed to “show what they know” by interacting with a game rather than taking a more traditional test or completing a worksheet. The card game also allowed students to practice in a less traditional way.

Results

Qualitative and quantitative methods were used to collect data including: (1) post-tests, (2) lesson plan development, (3) student interviews to assess student perceptions of the lessons, and (4) classroom observations used to assess fidelity of implementation.

A posttest only quasi-experimental design was employed for this study. The mean posttest score of the experimental group (n=14) was 48.10 (SD=14.46), while the mean of the control group (n=15) was 40.00(SD=11.75). This mean is below mastery level for both groups. However, controlling for initial differences in academic achievement using Spring 2011 PASS scores, this difference is significant at the .05 level (p = .025). In addition, as expressed in a survey and interviews following the instructional sequence, over 90% of students in the experimental group expressed high positive motivation and engagement when using the iPod touch. The researchers also recorded the perceptions of one teacher and one student using the iPod touch technology which were subsequently shared in YouTube videos. The teacher’s perceptions form the basis for the suggestions below.

Conclusions

During the study’s implementation, the teachers and the university faculty-in-residence decided that some key changes would be necessary for the more successful use of the iPod touch in instruction. One change implemented by the teacher after the study was to use co-teaching in small groups so that students’ iPod use could be better monitored. Also, in the next study, the researchers will select apps for the students to use that are more challenging. We found that when students were allowed to choose they picked the easier levels of the available apps. Some students were also more interested in scoring points in the game and winning, rather than challenging themselves. Sharing was an issue with single player apps because the time-on-task for each student was not always equal. In later lessons, the students also will be provided with dry erase boards for those who need a hands-on computational aid.

Students were engaged in the learning when using the apps for practice and enrichment. The teacher for the experimental group stated, “You could have heard a pin drop in the classroom when the students were using apps.”

Considerations Prior to and After Implementation

This research project was the first time we used the iPod touches in the classroom. We offer the following recommendations related to planning and classroom management for teachers who want to use iPod touches or other new technology devices in their classrooms: (1) Research apps carefully to be sure they align with your standards. Some apps we looked at seemed promising but then were not on the appropriate level to meet the expectation of the standard; (2) Integrate the use of the iPod touches into your classroom management plan the way you would other learning materials. Have clear procedures for passing out and returning the iPods; (3) Provide students with a brief workshop to familiarize themselves with the iPod before beginning the first lesson. This step is especially important if your students don’t have these devices at home; (4) Create a task list for students to hold them accountable for the activities in the lesson. It is easy for them to get off task when using the iPod. For example, one of the first steps in one of the fraction games was to design a rocket. Initially, students spent more time designing the rocket than practicing their fractions! (5) Circulate around the room constantly to monitor your students. Co-teaching with small groups is an alternative solution for ensuring all students are on task and getting their questions answered; and (6) Access to the wireless network may be a problem in some areas of your room or building. We recommend you identify these areas in your school to reduce classroom disruption.

Last, in returning to UDL principles, it is important to develop assessments in advance of the lessons, especially for formative purposes. Students should have multiple opportunities to show what they have learned in a variety of ways. For example, have students write a journal reflection or record their reflection using the iPod voice recorder or camera/ movie app. Teachers should also maintain anecdotal records and/or create a checklist of key activities to record information about the learning of all students while circulating the room.

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